Chapter- 6

Matroid Theory

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Abstract

A matroid is a structure that generalizes the properties of Independence. Matroid were introduced by Whitney in 1935 to provide a unifying abstract treatment of dependence in linear Algebra and graph theory. There are several ways to define a Matroid, each relate to the concept of independence.This project will focus on definitions of matroid in terms of Independent sets, circuits, bases and rank function and also discuss about transversal matroids and matroid optimization.

Keywords: Matroid, Independent Set, Bases, Rank, Transversals, Greedy Algorithm

Introduction

Matroid were introduced by Whitney in 1935 to provide a unifying abstract treatment of dependence in linear algebra and graph theory. There are several ways to define a matroid, each relate to the concept of independence. A characteristic of matroid is that they can be defined on many different but equivalent ways.The Theory of matroids originated in Linear Algebra and

Graph Theory and has deep connections with many other areas including Field theory, matching theory, submodular optimization, lie combinatorics and total positivity.

Preliminaries

Basic Graph Theory

By a graph $G(V, E)$we mean a finite set of vertices $V$and a set of edges E..An edge that joins a vertex to itself is called a loop. Edges that join the same pair of distinct vertices are called parallel edges . A graph with no loops and parallel edges are called simple. A graph H is a subgraph of a graph G if $V(H)$ and $E(H)$are subsets of $V(G)$and $E(G)$ respectively. A subgraph H of G is called proper if either $V(H) \neq V(G)$ or $E(H) \neq E(G)$.A walk in a graph is a sequence $v_0\, e_1 v_1 e_2 \dots v_{k-1} e_k v_k$ of vertices and edges and each vertex or edge in the sequence, except v_k , is incident with its successor in the sequence. If the vertices and the edges in the walk are distinct, it is a path.

A graph is closed i.e., $v_0 = v_k$ then it is called a cycle. A graph which contains no cycles is called forest or acyclic. A graph is connected if for any $v, w \in V(G)$ there exists a $v - w$ path. A connected acyclic graph is a tree.

Basic Linear Algebra

A non-empty set V is said to be a vector space over a scalar field F together with operations, addition and scalar multiplication, if it satisfy the following axioms:

1. If $\alpha, \beta \in V$, then $\alpha + \beta \in V$
2. $\alpha + \beta = \beta + \alpha$,for every $\alpha, \beta \in V$ (commutativity)
3. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for every $\alpha, \beta, \gamma \in V$(associativity)
4. There exists $0 \in V$ such that $\alpha + 0 = \alpha$ for every $\alpha \in V$ (Zero vector)
5. For every $\alpha \in V$ there exists $-\alpha \in V$ such that $\alpha + -\alpha = 0$(additive identity)
6. If $c \in F$ and $\alpha \in V$, then $c\alpha \in V$
7. $c(\alpha + \beta) = c\alpha + c\beta$ for every $c \in F$ and every $\alpha, \beta \in V$
8. $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$ for every $c_1, c_2 \in F$ and every $\alpha \in V$
9. $c_1(c_2\alpha) = (c_1 c_2)\alpha$ for every $c_1, c_2 \in F$ and every $\alpha \in V$
10. $1\alpha = \alpha$ for every $\alpha \in V$

Let V be a vector space over F, then a linear combination of vectors $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ in V is a vector $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\,\alpha_n$ for some scalars $c_1, c_2, \ldots, c_n$ in F .The vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ are said to be linearly independent if $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0 \Rightarrow c_i = 0\ \forall i$ Otherwise the vectors are said to be linearly dependent. Let V be a vector space over F, then the subset $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$of V is said to be a spanning set of V if for any $\alpha \in V$,$\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots. + c_n\alpha_n$ for some scalars $c_1, c_2, \ldots, c_n$in F.Let V be a vector space over F, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a subset of V ,then the Linear span $L(S)$ of S is given by $L(S) = \{\alpha \in V, \alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n, c_i \in F\}$. A subset S if a vector space V over F, is a basis for V if it is linearly independent and spans V

Basic Transversal Theory

Let E be a finite set and $S = (S_1, S_2, \ldots, S_m)$ be a family of non-empty subsets of E. A transversal of S is a set of m distinct elements of E, one chosen from each of subsets S_i such that a partial transversal of S is a Transversal of some subfamily of S.An equivalent way of graphically representing transversals is through bipartite graph.So the edges of the bipartite graph represent the membership of the elements in E to the subsets in S.A matching M $\subseteq E(G)$ in a graph G is a set of non-adjacent edges, while we say that M is a matching of some $U \subseteq V(G)$ if every vertex in U is an end-vertex of an edge in M. We can see that any matching in a bipartite graph of a set system corresponds to a partial transversal, and any matching of S corresponds to a transversal.

Independent Sets And Circuits

Independent Sets

A matroid M is an ordered pair (E, I) consisting of a finite set E and a collection I of subsets of E having the following three properties:

(I1)$\phi \in I$

(I2) If $I_1 \in I$and $I_2 \subseteq I$ then $I_2 \in I$

(I3) If $I_1, I_2 \in I$ and $|I_1| > |I_2|$ then there exist $i \in I_1 - I_2$ such that $I_2 \cup \{i\} \in I$.

The members of I are the independent sets of M and E is the ground set of M.A subset of E that is not in I is called dependent .

Proposition

Let E be the set of column labels of an $m \times n$ matrix A over a field F and let I be the set of subsets X of E for which the multiset of columns labelled by X is a set and is linearly independent in the vector space $V(m, F)$. Then (E, I) is a matroid.

Example

Let A be the following matrix over the field R of real numbers

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

If we denote the columns as 1, 2, 3, 4, 5 in order, then $E = \{1, 2, 3, 4, 5\}$ and

$I = \{\phi, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\}$

Thus the dependent sets of this matroid is,

$\{\{3\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}\} \cup \{X \subseteq E : |X| \geq 3\}$

Circuits

A circuit in a matroid M is a minimal dependent set i.e., A dependent set whose proper subsets are all independent.

We shall denote the set of circuits of M by C or $C(M)$. Once I has been specified, C can be determined and vice versa. The members of I are those subsets of E that contains no member of C.

Thus a matroid is uniquely determined by its set C of circuits. Clearly

(C1) $\phi \notin I$

(C2) If C_1 and C_2 are members of C and $C_1 \subseteq C_2$ then $C_1 = C_2$

(C3) If C_1 and C_2 are distinct members of C and $e \in C_1 \cap C_2$ then there is a member C_3 of C such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Theorem

Let E be a set and C be a collection of subsets of E satisfying (C1)-(C3). Let I be the collection of subsets of E that contain no member of C. Then (E, I) is a matroid having C as its collection of circuits.

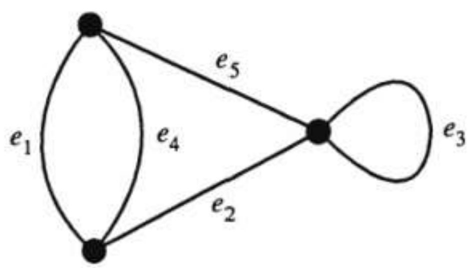
Proposition

Let E be the set of edges of a graph G and C be the set of edge cycles of . Then C is the set of circuits of a matroid on E.

Example

Let G be the graph shown below and let $M = M(G), E(M) = \{e_1, e_2, e_3, e_4, e_5\}$ and $C = \{\{e_3\}, \{e_1, e_4\}, \{e_1, e_2, e_5\}, \{e_4, e_2, e_5\}\}$.

Comparing M with the matroid $M[A]$ in the first example, we see that under the bijection ψ from {1, 2, 3, 4, 5} to $\{e_1, e_2, e_3, e_4, e_5\}$ defined by $\psi(i) = e_{i,}$a set X is a circuit in $M[A]$ if and only if $\psi(x)$ is a circuit in M. Equivalently, a set Y is independent in $M[A]$ and M have the same structure or are isomorphic.



Matroid that is isomorphic to the cycle matroid of a graph is called graphic matroid**.**

Formally two matroids $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ are isomorphic , written $\boldsymbol{M_1} \cong \boldsymbol{M_2}$ if there is a bijection $\boldsymbol{\psi}$ from $\boldsymbol{E}(\boldsymbol{M_1})$ to $\boldsymbol{E}(\boldsymbol{M_2})$ such that, for all X ⊆ $\boldsymbol{E}(\boldsymbol{M_1})$,the set $\boldsymbol{\Psi}(\boldsymbol{X})$ is independent in $\boldsymbol{M_2}$ if and only if $\boldsymbol{X}$ is independent in $\boldsymbol{M_1}$.

We call such a bijection Ψ an isomorphism from M_1 to M_2.

Bases And Rank

Bases

A basis or a base of M is the maximal independent set in M.If M is a matroid and B is its collection of bases,then

(B1) B is non-empty.

(B2) If B_1 and B_2 are members of B and $x \in B_1 - B_2$, then there is an element y of $B_2 - B_1$; $(B_1 - x) \cup y \in B$.(Basis exchange axiom)

Lemma : All the members of B have the same cardinality.

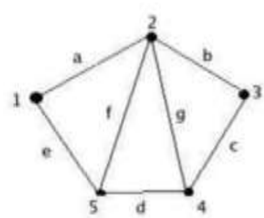
Theorem

Let E be a set and B be a collection of subsets of E satisfying (B1) and (B2).Let I be the collection of subsets of E that are contained in some member of B. Then (E, I) is a matroid having B as its collection of bases.

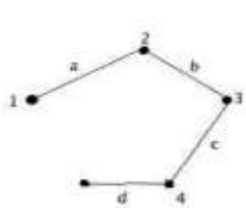
Corollary : Let B be a basis of a matroid M. If $e \in E(M) - B$, then $B \cup e$ contains a unique circuit $C(e, B)$.

Example

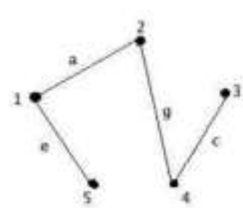
For graphic matroids, we will take the base of our matroid to be a spanning tree of graph G. Let G be the graph shown below



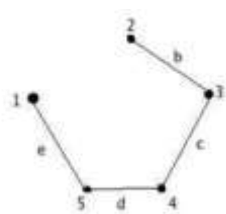
Then the bases are,{a, b, c, d}, {a, e, d, c}, {b, c, d, e}, {b, a, d, e}, {c, b, a, e}, {c, b, f, e}, {c, d, f, a},{c, g, a, e}, {c, g, f, e}.Clearly (B1) is satisfied. We can now demonstrate (B2);If we choose B_1 ={a, b, c, d} and B_2= {c, g, a, e}, then we can see the spanning trees of B_1 and B_2 in the following figures .Each spanning tree has 5 vertices and 4 edges.We can demonstrate (B2) by removing an element {a} from B_1 and then there exist an element in B_2 such that a new base is created ,B_3= (B_1\{a}) ∪ {e}).Figure 3.3 shows the new base B_2



3.1 Spanning Tree of B_1



3.2 Spanning Tree of B_2



3.3 Spanning Tree of B_3

Rank

We begin by defining a fundamental and very natural matroid construction .Let M be the matroid (E, I) and suppose that $X \subseteq E$. Let $I|X$ be $\{I \subseteq X : I \in I\}$.Then the pair $(X, I|X)$ is a matroid.We call this matroid the restriction of M to X or the deletion of $E - X$ from M.It is denoted by $M|X$ or $M|(E - X)$.

Definition

Rank $r(X)$ of X to be the cardinality of a basis B of $M|X$ and call such a set B a basis of X.Clearly the function r ,the rank function of M ,maps 2^E into the set of non negative integers.

R has the following properties:

(R1) If $X \subseteq$,then $0 \leq r(X) \leq |X|$

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

R(3) If X and Y are subsets of E,then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$

Lemma

Let E be a set and r be a function on 2^E satisfying (R2) and (R3).If X and Y are subsets of E such that $r(X \cup Y) = r(X)$ for all y in $Y - X$, then $r(X \cup Y) = r(X)$.

Theorem

Let E be a set and r be a function that maps 2^E into the set of non-negative Integers and satisfies (R1)-(R3).Let I be the collection of subsets X of E for which $r(X) = |X|$. Then (E, I) is a matroid having rank function r.

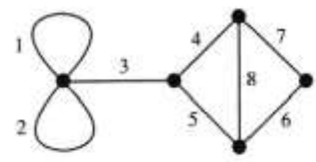
Remark

Let $M = M(G)$ where G is connected graph. Then a basis of $M(G)$ is the set of edges of a spanning tree of G. It is well known that for a tree T, $|V(T)| = |E(T)| + 1$.

Hence, $r(M) = |V(G)| - 1$

Example

Let $M = M(G)$ where G is the graph shown below. Then as G is connected



$R(M) = |V(G)| - 1 = 4$

If $X = \{4,5,6,7,8\}$, then a basis for $M|X$ is $\{4,5,6\}$, so

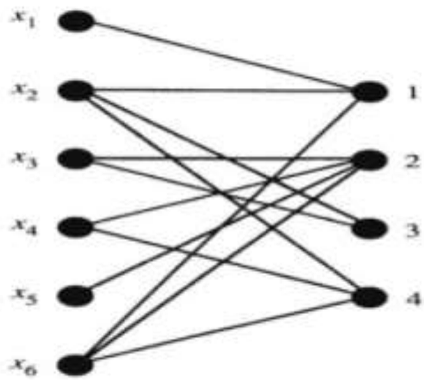
$r(\{4,5,6,7,8\}) = 3$

Transversal Matroids

For a finite set S, a family of subsets of S is a finite sequence $(A_1, A_2, \ldots, A_m)$ such that $A_j \subseteq S$ for all j in $\{1,2,\ldots,m\}$ Note that the terms of this sequence,the members of the family, need not be distinct. If $J = \{1,2,..,m\}$,We shall frequently abbreviate $\{A_1, A_2, \ldots, A_m\}$ as $(A_j: j \in J)$. A transversal or system of distinct representatives of $\{A_1, A_2, \ldots, A_m\}$ is a subset $\{e_1, e_2, \ldots, e_m\}$ of S such that $e_j \in A_j$.For all j in J, and $e_1, e_2, \ldots, e_m$ are distinct. Equivalently , T is a transversal of $(A_j: j \in J)$ if there is a bijection $\psi : J \to T$ such that $\psi(j) \in A_j$for all j in J.If $X \subseteq S$, then X is a partial transversal of $(A_j: j \in J)$ if X is a transversal of $(A_j: j \in K)$ for some subset K of J. In the special case that $(A_1, A_2, \ldots, A_m)$ is a partition π of S, the set of partial transversal of A coincides with the set of independent sets of the partition Matroid M_π. The main result we are discussing in this chapter is that, for all families A of subsets of S, the set of all partial transversals of A is the set of independent sets of a matroid on S.Another way to view partial transversals uses the idea of a matching in a bipartite graph.

Example

Let $S = \{x_1, x_2, \ldots, x_6\}$ and $A = \{A_1, A_2, A_3, A_4\}$ where $A_1 = \{x_1, x_2, x_6\}, A_2 = \{x_3, x_4, x_5, x_6\}$, $A_3 = \{x_2, x_3\}$, and $A_4 = \{x_2, x_4, x_6\}$. Then the bipartite graph Δ[A] is as shown in Figure.



The set $\{x_1, x_2, x_3, x_4\}$ is a transversal of A. To check this, one needs only check that $\{x_1 1, x_4 2, x_3 3, x_2 4\}$ is a matching in $\Delta[A]$. Similarly as,$\{x_6 1, x_2 3, x_4 3\}$ is a matching in $\Delta[A]$ the set $\{x_6, x_2, x_4\}$ is a partial transversal of A. Clearly A has many other partial transversals.

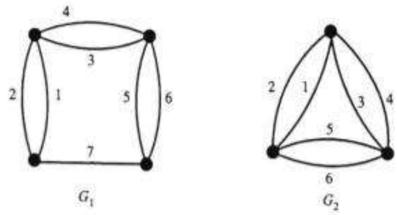
Theorem

Let A be a family $\{A_1, A_2, \ldots, A_m\}$ of subsets of a set S.Let I be the set of partial transversals of A. Then I is the collection of independent sets of a matroid on S.

Example

Let G_1 and G_2 be the graphs shown below.Let $A_1 = \{1, 2, 7\}$ $A_2 = \{3, 4, 7\}$ and $A_3 = \{5, 6, 7\}$. Then, for $A = (A_1, A_2, A_3)$ and $S = \{1, 2, \ldots, 7\}$,then $M[A] = M(G1)$.

In contrast, $M(G2)$ is not transversal. To show this, assume that $M(G2) = M[A']$ for some family A' of subsets of {1, 2, 3, 4, 5, 6} As {1} and {2} are independent but {1, 2} is dependent, there is a unique member, say $A_1{}'$ of A meeting {1, 2}. Moreover,$A_1{}'$ contains both 1 and 2. Similarly, A' has a unique member $A_2{}'$ meeting {3, 4} and a unique member $A_3{}'$ meeting {5, 6} and these members contain {3, 4} and {5, 6} respectively.As {1, 3}, {1, 5} and {3, 5} must be partial transversals of A',the sets $A_1{}', A_2{}'$ and $A_3{}'$ are distinct.This implies that {1, 3, 5} is a partial transversal of A'; a contradiction. We conclude that $M(G2)$ is indeed non-transversal.



Greedy Algorithm

Let G be a connected graph and let w be a function from $E(G)$into R.We call w a weight function on G and, for all $X \subseteq E(G)$, we define the weight $w(x)$ of X to be $\sum_{x \in X} w(x)$.

The greedy algorithm for the pair (I, w) proceeds as follows:

1. Set $X_0 = \phi$ and $j = 0$
2. If $E - X_j$ contains an element e such that $X_j \cup e \in I$ choose such an element e_{j+1} of maximum weight, let $X_{j+1} = X_k \cup e_{j+1}$and go to (3)
   Otherwise, let $B_G = X_j$jand go to (4).
3. Add 1 to j and go to (2).
4. Stop

Theorem

Let I be a collection of subsets of a set E.Then (E, I) is a matroid if and only if I has the following properties:

(I1) $\phi \in I$

(I2) If $I \in I$ and $I' \subseteq I$, then $I' \in I$.

(G) For all weight functions $w : E \rightarrow R$ the greedy algorithm produces a maximal member of I of maximum weight.

Conclusion

The theory of matroids has its origins in graph theory and linear algebra, and its most successful applications in the past have been jn the areas of combinatorial optimization and network theory.

Recently, however, there has been a flurry of new applications of this theory in the fields of information and coding theory. Its applications extend to diverse fields such as computer science, operation research, electrical engineering and more.

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