Chapter- 19

Zero Divisor Graph of a Commutative Ring

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Abstract
This paper investigates the structural and combinatorial properties of zero divisor graphs associated with commutative rings. Specifically, we study the domination number, total domination number, and co-total domination number of zero divisor graphs and provide a detailed analysis of their algebraic implications. We derive properties such as the radius and center of zero divisor graphs, classify specific cases where the graph forms a star graph or a complete bipartite graph, and establish domination-related parameters for various classes of commutative rings, including Z_n and $Z_p \times Z_q$ where *n, p, q* are integers or primes. These findings are supported with proofs and applications that extend existing research in this domain.

Keywords: Zero divisor graph, Domination number, Total domination number, Co-total domination number

1. Introduction

Let R be a commutative ring with unity, and let $Z(R)$ denote its set of zero divisors. The zero divisor graph $\Gamma(R)$ is a simple graph associated with R whose vertex set is $Z(R)^* = Z(R) \setminus \{0\}$, the set of all nonzero zero divisors of R. Two vertices x and y in $Z(R)^*$ are adjacent if $xy = 0$. If $\Gamma(R)$ is the empty graph, R is an integral domain.

The concept of the zero divisor graph was introduced by I. Beck in the context of graph colorings. This work was extended by D. D.

Anderson and M. Naseer. For all elements of *R*, we define $\Gamma_0(R)$, where vertices *x* and *y* are adjacent if *xy* = 0, including 000 as a universal vertex. Thus, *Γ(R)* is an induced subgraph of $\Gamma_0(R)$.

This study focuses on properties such as the center, radius, domination number, total domination number, and co-total domination number of zero divisor graphs and explores their algebraic implications.

2. Properties of Zero Divisor Graphs

2.1 Central Sets and Radius of Γ(R)\Gamma(R)Γ(R)

For a connected graph *G*, let *d(u,v)* denote the distance between vertices *u* and *v*. The eccentricity *e(v)* of a vertex *v* is defined as the maximum distance from *v* to any other vertex. The radius of *G*, *rad(G)*, is the minimum eccentricity among all vertices, and the set of vertices with minimal eccentricity forms the center of *G*.

- **Equation:**

$$rad(G)= \min_{v\in V(G)} e(v), \quad e(v) = \max_{u\in V(G)} d(u,v)$$

rad(G)=v∈V(G)mine(v),e(v)=u∈V(G)maxd(u,v).

- **Example:** The radius and center of $\Gamma(Z_n[i])$, the graph of Gaussian integers modulo *n*, are analyzed.

2.2 Domination Number

The domination number of a graph *Γ(R)* is the minimum number of vertices in a set $D \subseteq V(G)$ such that every vertex *v* *V(G)* is either in *D* or adjacent to some vertex in *D*.

- Definition: $\gamma(G) = \min\{ |D| : D \text{ is a dominating set}\}$

3. Results and Discussion

3.1 Theorem: Let *R* be a commutative ring (not necessarily finite). The graph *Γ(R)* is always connected, with diameter $diam(\Gamma(R)) \leq 3$

Proof: Consider *x, y* *Γ(R)*, with $x \neq y$.

If *xy* = 0, then *d (x, y)* =1.

If $xy \neq 0$:

- If $x^2 = 0$ and $y^2 = 0$, then $x - xy - y$ forms a path of length 2, so $d(x, y) = 2$.

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- If $x^2 = 0$ and $y^2 \neq 0$, there exists $b \neq y$ such that $by = 0$. If $bx = 0$, then $x - b - y$ forms a path of length 2. Otherwise, $x - ab - y$ forms a path of length 3.

In all cases, $d(x, y) \leq 3$, hence $diam(\Gamma(R)) \leq 3$.

3.2 Theorem: If $n = 2p$, where p is an odd prime, the nonzero zero divisor graph is a star graph $K_{1,p-1}$, which is also a complete bipartite graph.

Proof: Let $n = 2p$, where p is an odd prime, and consider the commutative ring $R = Z_n$. The set of nonzero zero divisors of R is given by

$Z(R)^* = \{x \quad Z_n : x \neq 0$ and $xy \equiv 0$ (mod n) for some $y \neq 0\}$

The vertex set of $\Gamma(R)$ is:

$V(\Gamma(R)) = Z(R)^* = \{2, 4, 6, \ldots, 2(p-1), p\}$,

where $\{2, 4, 6, \ldots, 2(p-1), p\}$ are the even integers less than n, and p is the odd prime divisor of n.

The edge set is defined as:

$E(\Gamma(R)) = \{(x, y) \quad Z(R)^* \times Z(R)^* : xy \equiv 0$ (mod n)$\}$.

The vertex p is adjacent to all even integers $\{2, 4, 6, \ldots, 2(p-1), p\}$ and no two even integers are adjacent to each other.

This structure forms a star graph, with p as the central vertex and $\{2, 4, 6, \ldots, 2(p-1), p\}$ as the leaves.

The graph can be partitioned into two subsets:

$X = \{p\}$, $\qquad Y = \{2, 4, 6, \ldots, 2(p-1)\}$.

Then the vertex $p \quad X$ is adjacent to every vertex in Y and There are no edges within X or Y.

Hence, $\Gamma(R)$ is isomorphic to the complete bipartite graph $K_{1,p-1}$.

3.3 Theorem: If $n = p^\alpha$, where p is an odd prime and $\alpha > 2$, the domination number $\gamma(\Gamma(Z_n)) = 1$

Proof: Here vertex set $Z(R)^* = \{p, 2p, \ldots, p^{\alpha-1}\}$ and vertex $p^{\alpha-1}$ is adjacent to all other vertex of $\Gamma(Z_n)$. Hence, $D = \{p^{\alpha-1}\}$ is a minimum dominating set. It is clear that $\{p^{\alpha-1}\}$ is minimum dominating set with cardinality 1. Therefore, the domination number of nonzero zero divisor graph and hence $\Gamma(Z_n) = 1$.

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3.4 Theorem: For $\Gamma(Z_p\times Z_q)$, where p and q are distinct primes: $\gamma_t(\Gamma(Z_p\times Z_q)) = 2$.

Proof: Vertex Set of $\Gamma(Z_p\times Z_q)$ is

$V\,\Gamma(Z_p\times Z_q)$ ={ (0, 1), (0, 2), …,(0, q−1), (1, 0), (2, 0), …, (p−1, 0)}.

This represents $q-1$ vertices of the form $(0, x)$ (where x ∈ Z^*_q and $p-1$ vertices of the form $(y, 0)$ (where y ∈ Z^*p). The graph is bipartite, with two disjoint subsets of vertices:

V_1 = {(1, 0), (2, 0), …, (p−1, 0)}, V_2 = {(0, 1), (0, 2), …, (0, q−1)}.

A total dominating set D_t is a subset of $V(\Gamma)$ such that every vertex in $V(\Gamma)$ is adjacent to at least one vertex in D_t.

1. Let us select one vertex from V_1, say (1, 0) and one vertex from V_2, say (0, 1).
2. The vertex (1, 0) dominates all vertices in V_2 and (0, 1) dominates all vertices in V_1.
3. Together, {(1, 0), (0, 1)} dominates all vertices of $\Gamma(Z_p\times Z_q)$.

Thus, the total domination number is: $\gamma_t\,\Gamma(Z_p\times Z_q) = 2$.

4. Conclusion

In this paper, we analysed the zero divisor graph Γ(R) for various commutative rings R, focusing on its domination number and total domination number. Our results demonstrate the following key findings:

For n=2p, where p is an odd prime, the nonzero zero divisor graph forms a star graph $K_{1,p-1}$, which is also a complete bipartite graph.

When $n=p^{\alpha}$, where p is an odd prime and $\alpha > 2$, the domination number $\gamma(\Gamma(Z_n)) = 1$.

For $\Gamma(Z_p\times Z_q)$, where p and q are distinct primes, the total domination number $\gamma_t(\Gamma(Zp\times Zq))= 2$.

These results not only refine the structural understanding of zero divisor graphs but also establish new connections between graph-theoretic parameters and algebraic properties of rings. The classifications and proofs presented contribute to ongoing research in algebraic graph theory, offering insights and tools for future exploration of domination-related properties in graphs derived from algebraic structures.

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