EXTORIAL FUNCTION AND ITS APPLICATIONS USING DELTA OPERATORS

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Chapter 1

Introduction

1.1 Difference Operators and Equations

A difference equation is an equation that contains sequence differences. There are various types of difference equations namely ordinary, delay, advanced, neutral, quasilinear, half linear, etc. These equations occur in numerous settings and forms, both in mathematics itself and its applications to Biology, Computer Science, Digital Signal Processing, Economics, Statistics and other fields.

The theory of difference equations, the methods used and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. In fact, in the last 15 years, the proliferation of the subject has been witnessed by hundreds of research articles, several monographs, many international conferences and numerous special sessions.

1.2 Difference Equation and its Solution

In numerical integration of differential equations a standard approach is to replace it by a suitable difference equation whose solution can be obtained in a stable manner and without troubles from round-off errors. There are two types of solutions for difference equations, one is numerical (or summation form) another one is closed form(or exact solution). However, the qualitative properties of solutions of the difference equations are quite different from the solutions of the corresponding differential equations. Solutions of several well known difference equations like Clairaut's, Euler's, Riccati's, Bernoulli's, Verhulst's, Duffing's, Mathieu's and Volterra's difference equations preserve most of the properties of the corresponding differential equations ([45]).

1.3 Growth of Generalized Difference Operators

The basic theory of difference equations is based on the difference operator Δ defined as $\Delta u(\kappa) = u(\kappa+1) - u(\kappa)$, where $\{u(\kappa)\}$ is a sequence or a function of κ of numbers. Many authors ([45],[47]) have suggested the definition of generalized difference operator Δ_ℓ on real valued function u defined on $\mathbb{R} = (-\infty, \infty)$ as

$$\Delta_\ell u(\kappa) = u(\kappa + \ell) - u(\kappa), \ \kappa \in \mathbb{R}, \ \ell > 0. \tag{1.1}$$

E. Thandapani, M.Maria Susai Manuel, G.B.A Xavier [42] considered the definition of Δ_ℓ as given in (1.1) and developed the theory of difference equations in a different direction. If there exists a function v such that $\Delta_\ell v(\kappa) = u(\kappa)$, then we call this function v as $\Delta_\ell^{-1} v$. Hence, for $\kappa \in \mathbb{R} = \bigcup_{0 \le j < \ell} \mathbb{N}_\ell(j)$,

$$\text{if } \Delta_\ell v(\kappa) = u(\kappa), \text{then } v(\kappa) = \Delta_\ell^{-1} u(\kappa) + c_j, \tag{1.2}$$

where c_j is constant for all κ in each $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \ldots\}$, $j = \kappa - [\frac{\kappa}{\ell}]\ell$.

In 1989, Miller and Rose introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. In 1984, Jerzy Popenda [27] introduced a particular type of difference operator on u as $\Delta_\alpha u(\kappa) = u(\kappa + 1) - \alpha u(\kappa)$, In 2011, M.Maria Susai Manuel, et.al, [34] extended the operator Δ_α to generalized $\alpha-$ difference operator as $\underset{\alpha(\ell)}{\Delta} v(\kappa) = v(\kappa + \ell) - \alpha v(\kappa)$ for real valued function v. In 2014, the authors in [6] have applied the q-difference operator defined by $\Delta_q v(\kappa) = v(q\kappa) - v(\kappa)$ and delta operator $\underset{\kappa(\ell)}{\Delta}$ with variable coefficients defined by $\underset{\kappa(\ell)}{\Delta} v(\kappa) = v(\kappa + \ell) - \kappa v(\kappa)$, $\ell \neq 0 \in \mathbb{R}$.

Also the generalized difference operator with n-shift values $l = (\ell_1, \ell_2, \ell_3, ..., \ell_n) \neq 0$ on a real valued function $v : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\underset{(\ell)}{\Delta} v(\kappa) = v(\kappa_1 + \ell_1, \kappa_2 + \ell_2, ..., \kappa_n + \ell_n) - v(\kappa_1, \kappa_2, ..., \kappa_n). \tag{1.3}$$

This operator $\underset{(\ell)}{\Delta}$ becomes generalized partial difference operator if some $\ell_i = 0$. The equations involving $\underset{(\ell)}{\Delta}$ with atleast one $\ell_i = 0$ is called generalized partial difference equation. for one shift value, we take $\Delta_{(\ell)}$ as Δ_ℓ. By defining the inverse Δ_ℓ^{-1}, many interesting results on sum of partial sums of higher power of arithmetic and geometric functions and applications in numerical methods (see [42],[41]) are obtained. The difference operator defined in (1.1) becomes the usual difference operator Δ when $\ell = 1$. We obtain several results on factorial function and Riemann zeta factorial function by applying Δ_ℓ^{-1}.

1.4 Riemann Zeta Factorial Function

The Riemann zeta function $\zeta(s)$ has been studied in many different forms for centuries. The harmonic series, $\zeta(1)$, has been proven to be divergent as far back as the 14th century [55]. Leonhard Euler, a Swiss mathematician discovered a closed form expression in 18th century for the sum of the reciprocals of the squared integers i.e.$\zeta(2)$. He also generalized this result and found a closed form expression for $\zeta(2n)$ for $n \in N$ [56]. In the 19th century, the German mathematician Bernhard Riemann considered ζ as a complex function. He published his work in the paper "On the Number of Primes Less Than a Given Magnitude", which is one of the most influential works of modern mathematics [5]. The classical definition of Riemann zeta function is $\zeta(s) = \sum\limits_{k=0}^{\infty} \frac{1}{\kappa^s} = \sum\limits_{\kappa=0}^{\infty} \kappa^{-s}$. Here, we develop higher order Riemann zeta factorial function obtained by replacing polynomials into polynomial factorials. Several properties of higher order Riemann zeta factorial functions are derived by applying the difference operator having shift value ℓ. Some applications of difference operator and its equation can be found in [1, 6, 7, 27, 34].

1.5 Extorial Function and RL circuit

Here we introduce extorial function and obtain numerical and exact solutions of certain type of ℓ difference equations. When $\ell \to 0$ the extorial function becomes exponential function and ℓ difference equations become Differential equations. The newly defined ℓ-Extorial function is arrived by replacing the polynomial κ^n by factorial polynomial $\kappa_\ell^{(n)}$ in the exponential function e^κ. The formal definition of extorial function is defined by

$$e(\kappa_\ell^{(n)}) = 1 + \frac{\kappa_\ell^{(n)}}{1!} + \frac{\kappa_\ell^{(2n)}}{2!} + \frac{\kappa_\ell^{(3n)}}{3!} + \cdots + \infty, |\ell| < 1, \kappa \in \mathbb{R}, \tag{1.4}$$

where $|\ell| \leq 1$, $\kappa \in \mathbb{R}$ and $n \in \mathbb{Z}$.

The following identities are arrived from (1.4):

(i) $e(\kappa_0^{(1)})$=e^κ, (ii) $e(\kappa_{-1}(1)) = \infty$, (iii)$e((-\kappa)_{(-\ell)}^{(1)}) = \sum\limits_{r=0}^{\infty}(-1)^r \frac{\kappa_\ell^r}{r!}$,

(iv) $e((-\kappa)_\ell^{(1)}) = \sum\limits_{r=0}^{\infty}(-1)^r \frac{\kappa_{(-\ell)^{(r)}}}{r!}$, (v) $\Delta_\ell e(\kappa_\ell^{(1)}) = \ell e(\kappa_\ell^{(1)})$,

(vi) $e(\kappa_\ell^{(1)}) = \sum\limits_{r=0}^{a} \frac{\kappa_\ell^{(r)}}{r!}$ if $\kappa = n\ell$ For $\kappa_1, \kappa_2 \in \mathbb{R}$ and $\ell \in (0, 1)$, The additive properties of extorial function is given by

$$e\Big((\kappa_1+\kappa_2)_\ell^{(1)}\Big)=e\Big((\kappa_1)_\ell^{(1)}\Big)e\Big((\kappa_2)_\ell^{(1)}\Big). \tag{1.5}$$

The negative index extorial function is defined as

$$e(\kappa_\ell^{-(n)})=1+\frac{1}{1!}\frac{1}{\kappa_\ell^{(n)}}+\frac{1}{2!}\frac{1}{\kappa_\ell^{(2n)}}+\frac{1}{3!}\frac{1}{\kappa_\ell^{(3n)}}+\cdots\infty,\kappa_\ell^{(rn)}\neq 0. \tag{1.6}$$

and we derive

$e(1_{-1}^{(-1)})=\sum\limits_{r=0}^{\infty}\frac{1}{(r!)^2}$, $e(-1_1^{(-1)})=\sum\limits_{r=0}^{\infty}(-1)^r\frac{1}{(r!)^2}$

For $\kappa_\ell^{(rn)}\neq 0$, $n\in\mathbb{N}$, $|\ell|<1$ and $\kappa_\ell^{-(n)}=\frac{1}{\kappa_\ell^{(n)}}$, we have

$$\Delta_\ell e(\kappa_\ell^{-(n)})=\frac{-n\ell}{(\kappa+\ell)_\ell^{(n+1)}}e((\kappa-n\ell)_\ell^{-(n)}). \tag{1.7}$$

Note that $\kappa_\ell^{(-n)}$ is different from $\kappa_\ell^{-(n)}$ and here we use the notation $\kappa_\ell^{-(n)}$.

For positive κ and $\ell>0$, we have

$$e(-\kappa_\ell^{-(1)})=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!}\frac{1}{\kappa_\ell^{(r)}}.$$

For $\ell\in(-1,1)$ and $\kappa\in\mathbb{R}$, the n^{th} order ℓ-extorial function $e_n(\kappa_\ell)$ is defined

$$e_n(\kappa_\ell)=1+\frac{\kappa_\ell^{(n)}}{n!}+\frac{\kappa_\ell^{(2n)}}{(2n)!}+\frac{\kappa_\ell^{(3n)}}{(3n)!}+\cdots+\infty. \tag{1.8}$$

Note that $e_n(\kappa_\ell)$ is different from $e(\kappa_\ell^{(n)})$.

From the definition of extorial function, we obtain following identities.

For real κ, $\ell>0$ and $n\in\mathbb{N}$, we have

(i) $$e_n(-\kappa_\ell)=\begin{cases} e_n(\kappa_{(-\ell)}) & \text{if } n \text{ is even,}\\ 1-\frac{\kappa_{(-\ell)}^{(n)}}{n!}+\frac{\kappa_{(-\ell)}^{(2n)}}{(2n)!}-\frac{\kappa_{(-\ell)}^{(3n)}}{(3n)!}+\cdots & \text{if } n \text{ is odd,}\end{cases}$$

and

$$\text{(ii)} \qquad e_n(-\kappa_{(-\ell)}) = \begin{cases} e_n(\kappa_{(\ell)}) & \text{if } n \text{ is even,} \\ 1 - \dfrac{(\kappa)_\ell^{(n)}}{n!} + \dfrac{(\kappa)_\ell^{(2n)}}{2n!} - \dfrac{(\kappa)_\ell^{(3n)}}{3n!} + \cdots & \text{if } n \text{ is odd,} \end{cases}$$

(iii) $\Delta_\ell e_n(\kappa_\ell) = \ell \sum_{m=1}^{\infty} \frac{\kappa_\ell^{(mn-1)}}{(mn-1)!}$, $nm \neq 1$, (iv) $\Delta_\ell^n e_n(\kappa_\ell) = e_n(\kappa_\ell^{(1)})$

(v) $\Delta_\ell^m e_m(\kappa_\ell) = \ell^m e_m(\kappa_\ell)$ and (vi) $\Delta_\ell^{-m} e_m(\kappa_\ell) = \dfrac{e_m(\kappa_\ell)}{\ell^m}$, $\ell \in \mathbb{N}$.

The summation form of $e_n(k_\ell)$ is obtained as

$$(vii) \qquad e_{-n}(\kappa_\ell) = \sum_{r=0}^{\infty} \frac{1}{(rn)!} \frac{1}{\kappa_\ell^{(rn)}} \tag{1.9}$$

These concepts are newly arrived and these have been used for obtaining new formula and application in RL circuits.

1.6 Summary

This book consists of eight chapters. In the first chapter, we present necessary introduction on difference operator, difference equations, growth of difference operator and formation of research work.

Chapter 2 provides relation between ℓ-delta operator and shift operator E, conversion of polynomial into polynomial factorial and vice-verse. For example,

$$\Delta_\ell^r = \sum_{j=0}^{r} (-1)^j \frac{r!}{(r-j)j!} E^{\ell(r-j)} \tag{1.10}$$

and

$$\Delta_\ell^r(\kappa^n) = \sum_{j=0}^{r} (-1)^i \frac{r!}{(r-j)j!}[\kappa + \ell(r-j)]^n = \begin{cases} 0 \text{ if } r > n \\ n!\ell^n \text{ if } r = n. \end{cases} \tag{1.11}$$

are arrived.

If S_r^n and s_i^n are Stirling numbers of second and first kinds respectively, then

$$\kappa^n = \sum_{r=1}^{n} S_r^n \ell^{n-r} \kappa_\ell^{(r)} \tag{1.12}$$

and

$$\kappa_\ell^{(n)} = \sum_{r=1}^{n} s_r^n \ell^{n-r} \kappa^r \tag{1.13}$$

Here, we give a method and a table to find Bernoulli's polynomials using stirling numbers of first and second kinds.

Chapter 3 deals with the higher order delta inverse on real valued function u. Here we derive a main theorem by introducing J_1 set. Assume that J_1 be a subset of real numbers such $\kappa \in J_1$ implies $\kappa \pm 1 \in J_1$ and $f : J_1 \to \mathbb{R}$ be a real valued function. If $\Delta^{-r} u(\kappa)\Big|_{\kappa=0} = 0$ for $r = 1, 2, 3, \cdots n,\ n \in \mathbb{N}(1)$, then

$$(\Delta_a^{-n} u)(\kappa) = \frac{1}{(n-1)!} \sum_{s=a}^{\kappa-n} (\kappa - s - 1)^{(n-1)} u(s), \tag{1.14}$$

where $a, \kappa \in J_1$ is an such that $k - a - n \in \mathbb{N}(1) = \{1, 2, 3, \cdots\}$

Also, the corresponding result related to Δ_ℓ is obtained below:

If $\kappa > m\ell$, $0 < \ell < \infty$ and $u(\kappa - m\ell) = 0$, then

$$\Delta_\ell^{-n} u(\kappa)\Big|_{\kappa-m\ell}^{\kappa} = \sum_{r=0}^{m} \frac{(n+r)^{(n)}}{\ell^n} u(\kappa - \ell - r\ell), \tag{1.15}$$

where $(n+r)^{(n)}$ is a falling factorial.

In Chapter 4, we introduce new function called as extorial function and obtain extorial type solutions of higher order linear $\ell-$difference equations with constant coefficients. Consider the n^{th} order linear difference equation of the form

$$\left(a_n \frac{\Delta_\ell^n}{\ell^n} + a_{n-1}\frac{\Delta_\ell^{n-1}}{\ell^{n-1}} + \cdots + a_0\right) u(\kappa) = e_1(t\kappa)_{t\ell}, \tag{1.16}$$

where $a_i's$ for $i = 1, 2, 3, ..., n$ are constants. Now, for the homogenous equation

$$\left(a_n \frac{\Delta_\ell^n}{\ell^n} + a_{n-1}\frac{\Delta_\ell^{n-1}}{\ell^{n-1}} + \cdots + a_0\right) u(\kappa) = 0 \tag{1.17}$$

we try $u(\kappa) = e_1((m\kappa)_{(m\ell)})$ as solution of (1.17). Then we get

$$\left(a_n \frac{\Delta_\ell^n e_1((m\kappa)_{(m\ell)})}{\ell^n} + a_{n-1}\frac{\Delta_\ell^{n-1} e_1((m\kappa)_{(m\ell)})}{\ell^{n-1}} + \cdots + a_0 e_1((m\kappa)_{(m\ell)})\right) u(\kappa) = 0. \tag{1.18}$$

Since $\Delta_\ell e_1((m\kappa)_{(m\ell)}) = m\ell e_1((m\kappa)_{(m\ell)}), \Delta_\ell^2 e_1((m\kappa)_{(m\ell)}) = (m\ell)^2 e_1((m\kappa)_{(m\ell)})$, we find that

$\Delta_\ell^n e_1((m\kappa)_{(m\ell)}) = (m\ell)^n e_1((m\kappa)_{(m\ell)})$. Substituting the values in (1.18), we get

$\frac{a_n}{\ell^n}(m\ell)^n e_1((m\kappa)_{m\ell}) + \frac{a_{n_1}}{\ell^{n-1}}(m\ell)^{n-1} e_1((m\kappa)_{m\ell}) + \cdots + a_0 e_1((m\kappa)_{m\ell}) = 0,$

which gives

$$\left(\frac{a_n}{\ell^n}(m\ell)^n + \frac{a_{n_1}}{\ell^{n-1}}(m\ell)^{n-1} + \cdots + a_0\right) = 0. \tag{1.19}$$

The auxiliary equation for (1.19) is obtained as

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_0 = 0. \tag{1.20}$$

Therefore, if m is a root of (1.20), $e_1((m\kappa)_{(m\ell)})$ becomes a solution of (1.17).

The particular solution of (1.16) is obtained as

$$u(\kappa) = \frac{e_1((t\kappa)_{(t\ell)})}{a_n e_1((t\ell)-1)^n + a_{n-1}e_1((t\ell)-1)^{n-1} + \cdots + a_0}.$$

Case 1 : Suppose zeros are real and different,then the complementary function for (1.16) is $u(\kappa) = A_1 e_1(m_1\kappa)_{(m_1\ell)} + A_2 e_1(m_2\kappa)_{(m_2\ell)} + \cdots + A_n e_1(m_2\kappa)_{(m_n\ell)}$, where A_i's are constants for all i=0,1,2,$\cdots n$. Therefore the general solution of (1.16) is

$$\begin{aligned} u(\kappa) = &\left[A_1 e_1((m_1\kappa)_{(m_1\ell)}) + A_2 e_1((m_2\kappa)_{(m_2\ell)}) + \cdots + A_n e_1((m_n\kappa)_{(m\ell)})\right] \\ &+ \frac{e_1((t\kappa)_{(t\ell)})}{a_n\left[e_1(t\ell)_{t\ell} - 1\right]^n + a_{n-1}e_1[((t\ell)_{t\ell}) - 1]^{n-1} + \cdots + a_0}. \end{aligned} \tag{1.21}$$

Case 2 : Suppose the roots are real and same then the general solution of (1.16) is

$$\begin{aligned} u(\kappa) = &\left[A_n + A_{n-1}(m\kappa)^{(n-1)}_{(m\ell)} + A_{n-2}(m\kappa)^{(n-2)}_{(m\ell)} + \cdots + A_1(m\kappa)_(m\ell)^{(1)}\right] e_1((m\kappa)_{(m\ell)}) \\ &+ \frac{e_1((t\kappa)_{(t\ell)})}{a_n e_1[((t\ell)_{t\ell}) - 1]^n + a_{n-1}e_1[((t\ell)_{t\ell}) - 1]^{n-1} + \cdots + a_0}. \end{aligned} \tag{1.22}$$

Illustrative examples are given in the book.

Chapter 5 focuses on the fractional order Riemann zeta factorial function defined as

$$\zeta^n_\ell(\kappa, s) = \Delta_\ell^{-(n-1)}\zeta_\ell(\kappa, s). \tag{1.23}$$

This function is obtained by replacing polynomials into factorials in the harmonic series like extorial function. When $n = 2$, $s \geq 3$, $\ell > 0$ and $(\kappa - 2\ell)^{s-2}_\ell \neq 0$, then we have the second order Riemann zeta factorial function as

$$\zeta_\ell^2(\kappa,s)=\sum_{t=0}^{\infty}\frac{(t+1)_1^{(1)}}{(\kappa+t\ell)_\ell^{(s)}}=\frac{1}{\ell^2(s-1)_1^{(2)}(\kappa-2\ell)_\ell^{(s-2)}}. \tag{1.24}$$

In general, m^{th} order Riemann zeta factorial function is expressed as

$$\zeta_\ell^m(\kappa,s)=\sum_{t=0}^{\infty}\frac{(t+(m-1))_1^{(m-1)}}{(m-1)!(\kappa+t\ell)_\ell^{(s)}}=\frac{1}{\ell^m(s-1)_1^{(m)}(\kappa-m\ell)_\ell^{(s-m)}}. \tag{1.25}$$

In Chapter 6, for each positive integer n and for $x\in(-\infty,\infty)$, the partial exponential function denoted as $e_n(x)$ is defined as

$$e_n(x)=1+\frac{x^n}{n!}+\frac{x^{2n}}{(2n)!}+\frac{x^{3n}}{(3n)!}+\cdots+\infty=\sum_{r=0}^{\infty}\frac{x^{rn}}{(rn)!}. \tag{1.26}$$

When $n=1$, (1.26) becomes exponential function.

Here, we find that the sub exponential function $e_n(x)$ given in (1.26)is a solution of the $(n-1)^{th}$ order linear non homogeneous differential equation

$$\frac{d^{n-1}}{dx^{n-1}}u(x)+\frac{d^{n-2}}{dx^{n-2}}u(x)+\cdots+\frac{d}{dx}u(x)+u(x)=e^x.$$

Chapter 7 is devoted for two dimensional nabla difference operator, defined by

$$\underset{(a)\ell}{\nabla}v(\kappa_1,\kappa_2)=v(\kappa_1,\kappa_2)-a_1v(\kappa_1+\ell_1,\kappa_2+\ell_2)-a_2v(\kappa_1+2\ell_1,\kappa_2+2\ell_2). \tag{1.27}$$

The inverse of nabla operator is defined as if there exists a function v such that

$$\underset{(a)\ell}{\nabla}v(\kappa_1,\kappa_2)=u(\kappa_1,\kappa_2)\Rightarrow v(\kappa_1,\kappa_2)=\underset{(a)\ell}{\overset{-1}{\nabla}}u(\kappa_1,\kappa_2)+c,, \tag{1.28}$$

where $a=(a_1,a_2)$ and $\ell=(\ell_1,\ell_2),(\kappa_1,\kappa_2)\in\mathbb{R}^2$ and c is an arbitrary constant.

Let $1-\sum_{j=1}^{2}a_je((j\ell_1)_{\ell_1})e((j\ell_2)_{\ell_2})\neq 0$. By (1.27) and (1.28), we have

$$\underset{(a)\ell}{\nabla}E(\kappa)=E(\kappa)-a_1E(\kappa+\ell)-a_2E(\kappa+2\ell), \tag{1.29}$$

where $E(\kappa) = e((\kappa_1)_{\ell_1})e((\kappa_2)_{\ell_2})$

Now, $\underset{(a)\ell}{\nabla} E(\kappa) = E(\kappa)[1 - a_1E(\ell) - a_2e(2(\ell_1)_{\ell_1})e(2(\ell_2)_{\ell_2})]$ yields

$$\overset{-1}{\underset{(a)\ell}{\nabla}} E(\kappa) = \frac{E(\kappa)}{1 - \sum\limits_{j=1}^{2} a_j e((j\ell_1)_{\ell_1})e((j\ell_2)_{\ell_2})}. \tag{1.30}$$

Let F_n denotes the n^{th} term of two parameter Fibonacci sequence . Let $v(\kappa_1, \kappa_2)$ be a solution of the equation $\underset{(a)\ell}{\nabla} v(\kappa_1, \kappa_2) = u(\kappa_1, \kappa_2), (\kappa_1, \kappa_2) \in \mathbb{R}^2$, then we obtain

$$v(\kappa_1, \kappa_2) - (F_nF_{n-1} + a_2F_{n-1})v(\kappa_1 + (n+1)\ell_1, \kappa_2 + (n+1)\ell_2)$$

$$- a_2F_nv(\kappa_1 + (n+2)\ell_1, \kappa_2 + (n+2)\ell_2) = \sum_{i=0}^{n} F_iu(\kappa_1 + i\ell_1, \kappa_2 + i\ell_2). \tag{1.31}$$

In the final Chapter 8, we give an application of extorial function in RL circuit and also we show that the extorial function $u(\kappa) = e_1((m\kappa)_{(m)})$ is a solution of second order linear difference equation

$$\Big(A\Delta^2 + B\Delta + C\Big)u(\kappa) = 0, \tag{1.32}$$

if m is a root of the auxiliary equation $Am^2 + Bm + C = 0$.

Let I_0 be initial value of $I(t)$. The de-energizing difference equation for $\nu = 1$, has a solution of the form

$$I(\kappa) = I_0e_1\left(\left(\frac{-R}{L}\kappa\right)_{(\frac{-R}{L})}\right). \tag{1.33}$$

For $\nu = 1$, the energizing difference equation has a solution

$$I(\kappa) = \frac{V}{\frac{L}{\ell}(e^{s\ell} - 1) + R} + I_0e_1\left(\left(\frac{-R}{L}\kappa\right)_{(\frac{-R}{L}\ell)}\right). \tag{1.34}$$

Also we apply extorial function to obtain solution of Heat filows.

Several applications are arrived in physical science using extorial functions.

Chapter 2

Stirling Numbres and Bernouilli's Polynomials

2.1 Introduction

Theory of generalized difference operator Δ_ℓ defined by $\Delta_\ell v(\kappa) = v(\kappa + \ell) - v(\kappa)$ is developed in [42]. Formulae for finding the sum of the n^{th} power of an arithmetic progression, sum of the products of n consecutive terms of an arithmetic progression, the sum of an arithmetic - geometric progression, qualitative properties of certain class of generalized difference equations are some of the applications of Δ_ℓ (see [42, 43]). Results developed in [42, 43] coincide with the corresponding results in [45, 47] when $\ell = 1$. Theory of $\Delta_{\pm\ell}$ and Generalized Bernoulli polynomials $B_n(\kappa, -\ell)$

for $(-\ell)$ with applications are established in [44]. Here, the theory of Δ_ℓ is extended to solutions of certain types of generalized difference equations, in particular generalized Clairaut's difference equation and generalized Euler difference equation. The generalized Bernoulli polynomials $B_n(\kappa, \ell)$ for $\ell, n \in \mathbb{N}(1)$ are solutions of the first order linear difference equation $v(\kappa+\ell)-v(\kappa) = n\kappa^{n-1}$, which yield the formula $a^n + (a+\ell)^n + (a+2\ell)^n + \ldots + (a+(\kappa-1)\ell)^n = \frac{1}{n+1}[B_{n+1}(a+\kappa\ell, \ell) - B_{n+1}(a, \ell)]$.

Here we use (i) $\mathbb{N}(a) = \{a, a+1, a+2, \ldots\}$, (ii) $\mathbb{N}_\ell(a) = a, a+\ell, a+2\ell, \cdots$, (iii) $\mathbb{Z}$ is the set of all integers, (iv) $c, c_0, c_1, c_2, \ldots$ are constants, (v) $\Gamma(\kappa) = \int_0^\infty e^{-t}t^{\kappa-1}dt$, and (vi) $\kappa^{(n)} = \kappa(\kappa-1)(\kappa-2)\ldots(\kappa-n+1), n \in \mathbb{N}(1)$.

2.2 Preliminaries

Here, we give basic definitions and some lemma's as Δ_ℓ operator and shift operator.

Definition 2.2.1. *[42] Let v be a real valued function defined on R. The generalized difference operator Δ_ℓ on $u(\kappa)$ is defined as*

$$\Delta_\ell(v(\kappa)) = v(\kappa+\ell) - v(\kappa). \tag{2.1}$$

For the shift operator E, the relation $E^\ell(v(\kappa)) = v(\kappa+\ell)$ gives

$$E^\ell = \Delta + 1 = (1+\Delta)^\ell. \tag{2.2}$$

The inverse of Δ_ℓ is defined as follows.

If $\Delta_\ell(u(\kappa)) = v(\kappa)$*, then* $u(\kappa) = \Delta_\ell^{-1}(v(\kappa)) + c$

Lemma 2.2.2. *[42] Let* $\ell, n, r \in \mathbb{N}(1)$*. Then the* r^{th} *order of* Δ_ℓ *is obtained as*

$$\Delta_\ell^r = \sum_{j=0}^{r} (-1)^j \frac{r!}{(r-j)!j!} E^{\ell(r-j)} \tag{2.3}$$

and

$$\Delta_\ell^r(\kappa^n) = \sum_{j=0}^{r} (-1)^i \frac{r!}{(r-j)!j!} [\kappa + \ell(r-j)]^n = \begin{cases} 0 \ \text{if } r > n \\ n!\ell^n \ \text{if } r = n. \end{cases} \tag{2.4}$$

Definition 2.2.3. *[42] If* $n \in \mathbb{N}(1)$*, then the* ℓ*-falling factorial* $\kappa_\ell^{(n)}$ *is defined by*

$$\kappa_\ell^{(n)} = \kappa(\kappa - \ell)(\kappa - 2\ell)\dots(\kappa - n\ell + \ell) \tag{2.5}$$

and for a real sequences $\{v(\kappa)\}_{\kappa \in \mathbb{Z}}$*, the closed* ℓ*-falling factorial on u(k) is*

$$[v(\kappa)]_\ell^{(n)} = v(\kappa)v(\kappa - \ell)\dots v(\kappa - n\ell + \ell).$$

Lemma 2.2.4. *[42] If* s_i^n *and* S_r^n *are Stirling numbers of first and second kinds respectively, then the relation between polynomial and factorial polynomials are*

$$\kappa^n = \sum_{r=1}^{n} S_r^n \ell^{n-r} \kappa_\ell^{(r)}, \tag{2.6}$$

and

$$\kappa_\ell^{(n)} = \sum_{r=1}^{n} s_r^n \ell^{n-r} \kappa^r. \tag{2.7}$$

Lemma 2.2.5. *[42] If* $\ell, m \in \mathbb{N}(1)$ *and* $\kappa \in \mathbb{N}(m\ell) = \{m\ell, (m+1)\ell, (m+2)\ell, \cdots\}$*. Then the* ℓ*- delta inverse of* $\kappa_\ell^{(m)}$ *is obtained as*

$$\Delta_\ell^{-1} \kappa_\ell^{(m)} = \frac{\kappa_\ell^{(m+1)}}{\ell(m+1)} + c_j \tag{2.8}$$

and the summation of ℓ- delta inverse of given function 'v' is

$$\Delta_\ell^{-1} v(\kappa\ell + i) = \sum_{r=0}^{\kappa-1} v(r\ell + j) + c_j,\ j = 0, 1, \ldots \ell - 1. \tag{2.9}$$

Lemma 2.2.6. *Let $n, \ell, m \in \mathbb{N}(1)$ and $n \geq m$. Then, if we assume $c_j = 0$ for all $\Delta_\ell^{-1}, \Delta_\ell^{-2}, \cdots, \Delta_\ell^{-n}$, then the m^{th} order ℓ- delta and its inverse of $\kappa_\ell^{(n)}$ are*

$$\Delta_\ell^m \kappa_\ell^{(n)} = (n\ell)_\ell^{(m)} \ell^m \kappa_\ell^{(n-m)}, n > m. \tag{2.10}$$

and

$$\Delta_\ell^{-m} \kappa_\ell^{(n-m)} = \frac{\kappa_\ell^{(n)}}{(n\ell)_\ell^{(m)}}.$$

Proof. From the definition of $\kappa_\ell^{(n)}$ and applying $\Delta_\ell \kappa_\ell^{(n)}$ we have

$\Delta_\ell \kappa_\ell^{(n)} = (\kappa + \ell)_\ell^{(n)} - \kappa_\ell^{(n)} = n\ell\kappa_\ell^{(n-1)}$

Again taking Δ_ℓ on both sides, we derive

$\Delta_\ell^2 \kappa_\ell^{(n)} = n\ell\Delta_\ell \kappa_\ell^{(n-1)} = (n\ell)(n-1)\ell\kappa_\ell^{(n-2)} = n^{(2)}\ell^2\kappa_\ell^{(n-2)}.$

Continuting this process upto Δ_ℓ^m we get the required result. □

Definition 2.2.7. *[43] If $m \in \mathbb{N}(1)$, then the equation of the form*

$$f(\kappa, v(\kappa), v(\kappa + \ell), v(\kappa + 2\ell), \ldots, v(\kappa + m\ell)) = 0$$

is called the generalized ℓ-difference equation.

Definition 2.2.8. *[47] Let $B_n(\kappa) = \kappa^n + b_1\kappa^{n-1} + b_2\kappa^{n-2} + \ldots + b_{n-1}\kappa + B_n(0)$ be an n^{th} degree are Bernoulli polynomials in κ. Then the generalized Bernoulli* *polynomials $B_n(\kappa,\ell)$ are defined as*

$$\ell B_n(\kappa,\ell) = \kappa^n + b_1\ell\kappa^{n-1} + b_2\ell^2\kappa^{n-2} + \ldots + b_{n-1}\ell^{n-1}\kappa + B_n(0)\ell^n, n \in \mathbb{N}(1)$$

Note that $B_n(\kappa,1) = B_n(\kappa)$, the Bernoulli's polynomial.

Example 2.2.9. *[47])Let $B_n(\kappa)$ be Bernoulli polynomials satisfying the equation $B_n(\kappa+1) - B_n(\kappa) = n\kappa^{n-1}$. Then $B_n(\kappa) = n\sum_{i=1}^{\kappa-1} i^{n-1} + B_n(0)$, where the Bernoulli's numbers $B_n(0)$ are obtained by equating the coefficients of λ^n in*

$$\left(1 + \frac{\lambda}{2!} + \frac{\lambda^3}{3!} + ...\right)^{-1} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} B_n(0)$$

and

$$\kappa^n = \frac{1}{n+1}\sum_{i=0}^{n} \frac{(n+1)!}{(n+1-i)!i!}. \tag{2.11}$$

Example 2.2.10. *[45] For the Euler second order linear difference equation $4(\kappa+1)\kappa\Delta^2 u(\kappa) + 4\kappa\Delta u(\kappa) + 9u(\kappa) = 0$, the polynomial $\sum_{j=0}^{n} a_j(\lambda)^{(j)} = 0$ reduces to $4\lambda^2 + 9 = 0$. Thus, $u_1(\kappa) = \frac{\Gamma\left(\kappa+\frac{3}{2}i\right)}{\Gamma(\kappa)}$ and $u_2(\kappa) = \frac{\Gamma\left(\kappa-\frac{3}{2}i\right)}{\Gamma(\kappa)}$*

are the solutions of the above Euler difference equation. However, since

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{x-1}\cos(y\ln t)\,dt + i\int_0^{\infty} e^{-t}t^{x-1}\sin(y\ln t)\,dt$$

it follows that,

$$u_1(\kappa) = \frac{1}{\Gamma(\kappa)}\int_0^{\infty} e^{-t}t^{\kappa-1}\cos\left(\frac{3}{2}\ln t\right)dt$$

and

$$u_2(\kappa) = \frac{1}{\Gamma(\kappa)} i\int_0^{\infty} e^{-t}t^{\kappa-1}\sin\left(\frac{3}{2}\ln t\right)dt$$

are linearly independent solutions of the Euler difference equation.

2.3 Bernoulli's Polynomials

In this section, we develop Bernoulli's polynomials by Stirling numbers of both first and second kinds and it is given table form.

Theorem 2.3.1. *If ℓ, m, n are positive integers such that $m \leq n$ and $\kappa \in \mathbb{N}_\ell(n\ell)$, then $v(\kappa) = \kappa$ is a solution of the generalized factorial difference equation*

$$\sum_{r=0}^{m}(-1)^r \frac{m!}{(m-r)!r!} u(\kappa + \ell - m - 1) = (n\ell)_\ell^{(m)} (v(\kappa))_\ell^{(n-m)} \tag{2.12}$$

where $(v(\kappa))_\ell^{(n)} = v(\kappa)u(\kappa - \ell)\cdots u(\kappa - n\ell - \ell)$.

Proof. Using the shift operator E, equation (2.12) can be expressed as

$$\left[\sum_{r=0}^{m}(-1)^r \frac{m!}{(m-r)!r!} E^{\ell(m-r)}\right] (v(\kappa))_\ell^{(n)} = (n)_\ell^{(m)}\, \ell^m\, (v(\kappa))_\ell^{(n-m)},$$

which yields

$$\left(E^\ell - 1\right)^m (v(\kappa))_\ell^{(n)} = (n)_\ell^{(m)}\, \ell^m\, (v(\kappa))_\ell^{(n-m)}. \tag{2.13}$$

Using (2.2) in (2.13), we obtain

$$\Delta_\ell^m\, (v(\kappa))_\ell^{(n)} = (n)_\ell^{(m)}\, \ell^m\, (v(\kappa))_\ell^{(n-m)}. \tag{2.14}$$

Now the proof follows from Lemma 2.2.6 and (2.14). □

Theorem 2.3.2. *If m and ℓ are positive integers, then any polynomial in κ of degree m with leading coefficient $\frac{c}{m!\ell^m}$ is a solution of the generalized difference equation*

$$\sum_{r=0}^{m}(-1)^r \frac{m!}{(m-r)!r!} v(\kappa + m\ell - r\ell) = c. \tag{2.15}$$

Proof. The equation (2.15) can be expressed as

$$\sum_{r=0}^{m}(-1)^r\frac{m!}{(m-r)!r!}E^{m\ell-r\ell}v(\kappa)=\left(\frac{c}{m!\ell^m}\right)m!\ell^m.$$

Using (2.2), it reduces to

$$\Delta_\ell^m v(\kappa)=\left(\frac{c}{m!\ell^m}\right)m!\ell^m. \tag{2.16}$$

From (2.4), we obtain

$$\Delta_\ell^m\left\{\frac{c}{m!\ell^m}\kappa^m+c_1\kappa^{m-1}+...+c_m\right\}=c. \tag{2.17}$$

Now, the proof follows from (2.16) and (2.17). □

Theorem 2.3.3. *Let* $\ell,m\in\mathbb{N}(1)$ *and* $0\le r\le m$. *Then*

$$v(\kappa)=\frac{a_0}{m!\ell^m}\kappa^m+a_1\kappa^{m-1}+a_2\kappa^{m-2}+...+a_m$$

is a solution of the generalized difference equation

$$\sum_{j=0}^{r}\frac{m!}{(m-r)!r!}v(\kappa+(m-j)\ell)$$

$$=\sum_{j=0}^{r}\frac{(-1)^{m-j+1}}{(m-j)!j!}m!\sum_{r=0}^{m}\frac{a_0}{m!\ell^m}(\kappa+j\ell)^{m-r}, \tag{2.18}$$

where a_j*'s are arbitrary constants.*

Proof. By defining $v(\kappa)=\frac{a_0}{m!\ell^m}\kappa^m+a_1\kappa^{m-1}+a_2\kappa^{m-2}+...+a_m$, (2.18) can be expressed as

$$\sum_{j=0}^{m}(-1)^j\sum_{j=0}^{r}\frac{m!}{(m-r)!r!}v(\kappa+(m-j)\ell)=a_0 \tag{2.19}$$

The proof now follows from Theorem 2.3.2. □

Theorem 2.3.4. *For $n \in \mathbb{N}(0)$, the generalized Bernoulli polynomial $B_{n+1}(\kappa,\ell)$ is a solution of the generalized difference equation*

$$v(\kappa+\ell)-v(\kappa)=(n+1)\kappa^n. \tag{2.20}$$

Furthermore, if $\ell, n \in \mathbb{N}(1)$ and s_j^n, S_j^n are Stirling numbers of first and second kinds, then we have

$$B_{n+1}(\kappa,\ell)=\sum_{j=1}^{n}\frac{n+1}{j+1}S_j^n s_j^{j+1}\ell^{n-1}\kappa+\sum_{r=1}^{n}\left(\sum_{j=r}^{n}\frac{n+1}{j+1}S_j^n s_{r+1}^{j+1}\right)\ell^{n-r-1}\kappa^{r+1}+B_{n+1}(0)\ell^n, \tag{2.21}$$

where the Bernoulli numbers $B_n(0)$ can be obtained by the recurrence relations

$$B_1(0)=1; B_n(0)=-\frac{1}{n}\sum_{j=0}^{n-1}B_{j+1}(0)\sum_{j=0}^{r}\frac{n!}{(n-r)!r!}, n \in \mathbb{N}(2). \tag{2.22}$$

Proof. By (2.7) the equation definition (2.2.1), (2.20) becomes

$$v(\kappa)=(n+1)\Delta_\ell^{-1}\left(\sum_{r=1}^{n}S_r^n\ell^{n-r}\kappa_\ell^{(r)}\right)+A, \tag{2.23}$$

where A is an arbitrary constant and $n \in \mathbb{N}(1)$.

Applying (2.8) in (2.23), we obtain

$$v(\kappa)=(n+1)\left(\sum_{r=1}^{n}S_r^n\ell^{n-r}\frac{\kappa_\ell^{(r+1)}}{\ell(r+1)}\right)+A.$$

By definition (2.2.8) and assuming $A=B_{n+1}(0)\ell^n$, we get

$$v(\kappa)=B_{n+1}(\kappa,\ell)=\sum_{r=1}^{n}\frac{n+1}{r+1}S_r^n\ell^{n-r-1}\kappa_\ell^{(r+1)}+\ell^nB_{n+1}(0). \tag{2.24}$$

Using (2.7) in (2.24), we obtain

$$\mathbb{B}_{n+1}(\kappa,\ell) = \sum_{r=1}^{n} \frac{n+1}{r+1}\ell^{n-r-1} S_r^n \sum_{j=1}^{r+1} s_j^{r+1}\ell^{r-1}\kappa^j + B_{n+1}(0)\ell^n,$$

which yields (2.21). Example (2.2.9) yields (2.22) and the proof now follows by definition (2.2.1). □

Corollary 2.3.5. *$v(\kappa) = \sum_{m=0}^{n} \frac{a_m}{m+1} B_{m+1}(\kappa,\ell)$ is a solution to the generalized difference equation $y_{\kappa+\ell} - v(\kappa) = \sum_{m=0}^{n} a_m \kappa^m, \kappa \in \mathbb{N}(0), \ell \in \mathbb{N}(1)$, where a_i's are given constants (assume $S_0^0 = 1$).*

The following example illustrates the Theorem 2.3.4.

Example 2.3.6. *Here we establish a method to find the generalized Bernoulli polynomials $B_n(\kappa,\ell)$ and obtain the generalized Bernoulli polynomials in κ for ℓ up to degree 10 which are solutions to $v(\kappa+\ell) - v(\kappa) = n\kappa^{n-1}, n = 0, 1, 2, \ldots 10$.*

Dividing $\sum_{j=0}^{r} \frac{n!}{(n-r)!r!}$ from the Pascal triangle for $\sum_{j=0}^{r} \frac{n!}{(n-r)!r!}$ and using (2.22), we obtain table (i) for getting Bernoulli's number.

(i) (Note that $B_n(0) = -$ sum of product of the number and shift number in the n^{th} row).

										$B_n(0)$
										1
									$\frac{1}{2}_{(1)}$	$\frac{-1}{2}$
								$\frac{1}{3}_{(1)}$	$1_{(\frac{-1}{2})}$	$\frac{1}{6}$
							$\frac{1}{4}_{(1)}$	$1_{(\frac{-1}{2})}$	$\frac{3}{2}_{(\frac{1}{6})}$	0
						$\frac{1}{5}_{(1)}$	$1_{(\frac{-1}{2})}$	$2_{(\frac{1}{6})}$	$2_{(0)}$	$\frac{-1}{30}$
					$\frac{1}{6}_{(1)}$	$1_{(\frac{-1}{2})}$	$\frac{5}{2}_{(\frac{1}{6})}$	$\frac{10}{3}_{(0)}$	$\frac{5}{2}_{(\frac{-1}{30})}$	0
				$\frac{1}{7}_{(1)}$	$1_{(\frac{-1}{2})}$	$3_{(\frac{1}{6})}$	$5_{(0)}$	$5_{(\frac{-1}{30})}$	$3_{(0)}$	$\frac{1}{42}$
			$\frac{1}{8}_{(1)}$	$1_{(\frac{-1}{2})}$	$\frac{7}{2}_{(\frac{1}{6})}$	$7_{(0)}$	$\frac{35}{4}_{(\frac{-1}{30})}$	$7_{(0)}$	$\frac{7}{2}_{(\frac{1}{42})}$	0
		$\frac{1}{9}_{(1)}$	$1_{(\frac{-1}{2})}$	$4_{(\frac{1}{6})}$	$\frac{28}{3}_{(0)}$	$14_{(\frac{-1}{30})}$	$14_{(0)}$	$\frac{28}{3}_{(\frac{1}{42})}$	$4_{(0)}$	$\frac{-1}{30}$
	$\frac{1}{10}_{(1)}$	$1_{(\frac{-1}{2})}$	$\frac{9}{2}_{(\frac{1}{6})}$	$12_{(0)}$	$21_{(\frac{-1}{30})}$	$\frac{126}{5}_{(0)}$	$21_{(\frac{1}{42})}$	$12_{(0)}$	$\frac{9}{2}_{(\frac{-1}{30})}$	0
$\frac{1}{11}_{(1)}$	$1_{(\frac{-1}{2})}$	$5_{(\frac{1}{6})}$	$15_{(0)}$	$15_{(\frac{-1}{30})}$	$42_{(0)}$	$42_{(\frac{1}{42})}$	$30_{(0)}$	$15_{(\frac{-1}{30})}$	$5_{(0)}$	$\frac{5}{66}$

table (i) Bernoulli's number

$S_i^{n+1} = S_{i-1}^n + iS_i^n$ generates table (ii).

1_1								
1_1	1_2							
1_1	3_2	1_3						
1_1	7_2	6_3	1_4					
1_1	15_2	25_3	10_4	1_5				
1_1	31_2	90_3	65_4	15_5	1_6			
1_1	63_2	301_3	350_4	140_5	21_6	1_7		
1_1	127_2	966_3	1701_4	1050_5	266_6	28_7	1_8	
1_1	255_2	3025_3	7770_4	6951_5	2646_6	462_7	36_8	1_9

table (ii) Stirling number of second kind

1_{-1}											
-1_{-2}	-1_{-2}		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2_{-3}	-3_{-3}	1_{-3}		$\frac{1}{3}$	1	$\frac{7}{3}$	5	$\frac{31}{8}$	21	$\frac{127}{3}$	85
-6	11	-6	1		$\frac{1}{4}$	$\frac{3}{2}$	$\frac{25}{4}$	$\frac{45}{2}$	$\frac{301}{4}$	$\frac{483}{2}$	$\frac{3025}{4}$
24	-50	35	-10	1		$\frac{1}{5}$	2	13	70	$\frac{1701}{5}$	1554
-120	274	-225	85	-15	1		$\frac{1}{6}$	$\frac{5}{2}$	$\frac{70}{3}$	175	$\frac{2317}{2}$
720	-1764	1624	-735	175	-21	1		$\frac{1}{7}$	3	38	378
-5040	13068	-13132	6769	-1960	322	-28	1		$\frac{1}{8}$	$\frac{7}{2}$	$\frac{231}{4}$
40320	-109584	118124	-67284	22449	-4536	546	-36	1		$\frac{1}{9}$	4
-362880	1026576	-1172700	723680	-269325	63273	-9450	870	-45	1		$\frac{1}{10}$
κ	κ^2	κ^3	κ^4	κ^5	κ^6	κ^7	κ^8	κ^9	κ^{10}		
0	$-\frac{3}{2}$	0	5	0	-7	0	$\frac{15}{2}$	-5	1	$B_{10}(\kappa)$	$\times 10$
$-\frac{3}{10}$	0	2	0	$-\frac{21}{5}$	0	6	$-\frac{9}{2}$	1	$B_9(\kappa)$	$\times 9$	
0	$\frac{2}{3}$	0	$-\frac{7}{3}$	0	$\frac{14}{3}$	-4	1	$B_8(\kappa)$	$\times 8$		
$\frac{1}{6}$	0	$-\frac{7}{6}$	0	$\frac{7}{2}$	$-\frac{7}{2}$	1	$B_7(\kappa)$	$\times 7$			
0	$-\frac{1}{2}$	0	$\frac{5}{2}$	-3	1	$B_6(\kappa)$	$\times 6$				
$-\frac{1}{6}$	0	$\frac{5}{3}$	$-\frac{5}{2}$	1	$B_5(\kappa)$	$\times 5$					
0	1	-2	1	$B_4(\kappa)$	$\times 4$						
$\frac{1}{2}$	$-\frac{3}{2}$	1	$B_3(\kappa)$	$\times 3$							
-1	1	$B_2(\kappa)$	$\times 2$								
1	$B_1(\kappa)$	$\times 1$									

table (iii) Bernoulli's polynomials

In table (iii), upper north west triangular values are the Stirling numbers of first kind having the relations $s_i^{n+1} = s_{i-1}^n - ns_i^n$ and upper north east triangular values $\dfrac{S_i^n}{i+1}$ are obtained from table (ii). $n\left\langle (s_1^2, s_1^3, ...s_1^n), \left(\frac{S_1^{n-1}}{2}, \frac{S_2^{n-1}}{3}, ...\frac{S_{n-1}^{n-1}}{n}\right)\right\rangle$ is the coefficient of $\kappa\ell^{n-2}$ of $B_n(\kappa,\ell)$, $n\left\langle \left(s_{j+1}^{j+1}, s_{j+1}^{j+2}, ..., s_{j+1}^n\right), \left(\frac{S_j^{n-1}}{j+1}, \frac{S_{j+1}^{n-1}}{j+2}, ..., \frac{S_{n-1}^{n-1}}{n}\right)\right\rangle$ is the coefficient of $\kappa^{j+1}\ell^{n-j+2}$ of $B_n(\kappa,\ell)$ for $j = 1, 2, \ldots, n$ and $n \in \mathbb{N}(2)$ where $\langle . \rangle$ denote the inner product. 1 is the coefficient of $\kappa\ell^{-1}$ of $B_1(\kappa,\ell)$. The values of $B_n(0)$ of table (i) and the lower triangular values (coefficients) of table (iii) generate the Bernoulli polynomials of degree up to 10 as given below.

$B_{10}(\kappa,\ell) = \frac{5}{66}\ell^9 - \frac{3}{2}\ell^7\kappa^2 + 5\ell^5\kappa^4 - 7\ell^3\kappa^6 + \frac{15}{2}\ell\kappa^8 - 5\kappa^9 + \ell^{-1}\kappa^{10}$

$B_9\ (\kappa,\ell) = -\frac{3}{10}\ell^7\kappa + 2\ell^5\kappa^3 - \frac{21}{5}\ell^3\kappa^5 + 6\ell\kappa^7 - \frac{9}{2}\kappa^8 + \ell^{-1}\kappa^9$

$B_8\ (\kappa,\ell) = \frac{1}{30}\ell^7 + \frac{2}{3}\ell^5\kappa^2 - \frac{7}{3}\ell^3\kappa^4 + \frac{14}{3}\ell\kappa^6 - 4\kappa^7 + \ell^{-1}\kappa^8$

$B_7\ (\kappa,\ell) = \frac{1}{6}\ell^5\kappa - \frac{7}{6}\ell^3\kappa^3 + \frac{7}{2}\ell\kappa^5 - \frac{7}{2}\kappa^6 + \ell^{-1}\kappa^7$

$B_6\ (\kappa,\ell) = \frac{1}{4}\ell^5 - \frac{1}{2}\ell^3\kappa^2 + \frac{5}{2}\ell\kappa^4 - 3\kappa^5 + \ell^{-1}\kappa^6$

$B_5\ (\kappa,\ell) = -\frac{1}{6}\ell^3\kappa + \frac{5}{3}\ell\kappa^3 - \frac{5}{2}\kappa^4 + \ell^{-1}\kappa^4$

$B_4\ (\kappa,\ell) = -\frac{1}{30}\ell^3 + \ell\kappa^2 - 2\kappa^3 + \ell^{-1}\kappa^4$

$B_3\ (\kappa,\ell) = \frac{1}{2}\ell\kappa - \frac{3}{2}\kappa^2 + \ell^{-1}\kappa^3$

$B_2\ (\kappa,\ell) = \frac{1}{6}\ell - \kappa + \ell^{-1}\kappa^2$

$B_1\ (\kappa,\ell) = -\frac{1}{2} + \ell^{-1}\kappa.$

The above polynomials $B_n(\kappa,\ell)$ are solutions of the generalized difference equations

$v(\kappa+\ell)-v(\kappa)=n\kappa^{n-1}$ for $n=10,9,\ldots,1$ respectively.

Corollary 2.3.7. *If $B_{n+1}(\kappa,\ell)$ is the generalized Bernoulli polynomial for ℓ in κ of degree $n+1$, then we have the relation*

$$a^n+(a+\ell)^n+(a+2\ell)^n+\ldots+(a+\kappa\ell-\ell)^n=\frac{1}{n+1}[B_{n+1}(a+\kappa\ell,\ell)-B_{n+1}(a,\ell)]. \quad (2.25)$$

Proof. From Theorem 2.3.4 and $\Delta_\ell B_{n+1}(\kappa,\ell)=(n+1)\kappa^n$, we find

$$\frac{1}{n+1}B_{n+1}(\kappa\ell+a,\ell)=\Delta_\ell^{-1}(\kappa\ell+a)^n.$$

Now, the proof follows by substituting $v(\kappa)=\kappa^n$ in (2.9).

The following example is an illustration of (2.25). □

Example 2.3.8. *Using (2.25) the sum of the 9^{th} powers of arithmetic progression with initial term $a=4$, common difference $\ell=3$ and last term 997 is obtained as $4^9+7^9+\ldots+997^9=\frac{1}{10}[B_{10}(1000,3)-B_{10}(4,3)]$, where $B_{10}(k,\ell)$ is given in the example 2.3.6.*

2.4 Generalized Clairaut's and Eulers Difference Equation

In the following, we present certain types of difference equations and its solutions. In particular we obtain solutions of discrete generalized clairaut's and euler difference equation using ℓ- delta theory.

Example 2.4.1. *(i) The equation* $v(\kappa+\ell) = (-1)^{\ell}\kappa^{(\ell)}(\kappa-1)^{(\ell)}v(\kappa)$,

has a solution $v(\kappa) = c(-1)^{\kappa}\Gamma(\kappa-\ell+1)\Gamma(\kappa-\ell), \Gamma(\kappa) = (\kappa-1)\Gamma(\kappa-1)$.

(ii) Since $v(\kappa+\ell) = \dfrac{c3^{\kappa+\ell}\Gamma(\kappa+1+1)}{[\Gamma(\kappa+1)]^2} = \dfrac{3^{\ell}(\kappa+1)^{(\ell)}}{[\kappa^{(\ell)}]^2}\dfrac{C3^{\kappa}\Gamma(\kappa+1-\ell+1)}{[\Gamma(\kappa-\ell+1)]^2}$.

The equation $v(\kappa+\ell) = 3^{\ell}\dfrac{(\kappa+1)^{(\ell)}}{[\kappa^{(\ell)}]^2}v(\kappa)$, *has a solution* $v(\kappa) = c3^{\kappa}\dfrac{(\kappa-\ell+1)}{\Gamma(\kappa-\ell+1)}$,

(iii) The solution $v(\kappa) = c\dfrac{\kappa-\ell}{\Gamma\left(\kappa+\dfrac{3}{2}-\ell\right)}$ *of the equation*

$v(\kappa+\ell) = \dfrac{\kappa^{(\ell)}}{(\kappa-1)^{(\ell)}\left(\kappa+\dfrac{1}{2}\right)^{(\ell)}}v(\kappa)$ *is obtained form*

$$v(\kappa) = c\frac{\Gamma(\kappa-\ell+1)}{\Gamma(\kappa-1-\ell+1)\Gamma\left(\kappa+\dfrac{1}{2}-\ell+1\right)}$$

(iv) The equation $(a\kappa+b)^{(\ell)}_{(a)}v(\kappa+\ell) + (c\kappa+d)^{(\ell)}_{(c)}v(\kappa) = 0, a \neq 0, c \neq 0$,

has a solution $v(\kappa) = c\dfrac{\left((-1)^{\frac{1}{\ell}}\dfrac{c}{a}\right)^{\kappa}\Gamma\left(\kappa+\dfrac{d}{c}-\ell+1\right)}{\Gamma\left(\kappa+\dfrac{b}{a}-\ell+1\right)}$,

since $v(\kappa+\ell) = -\dfrac{(c\kappa+d)^{(\ell)}_{(c)}}{(a\kappa+b)^{(\ell)}_{(a)}}v(\kappa) = \left((-1)^{\frac{1}{\ell}}\dfrac{c}{a}\right)^{\ell}\dfrac{\left(\kappa+\dfrac{d}{c}\right)^{(\ell)}}{\left(\kappa+\dfrac{b}{a}\right)^{(\ell)}}v(\kappa)$.

(v) The equation $v(\kappa+\ell) = \dfrac{c^{\ell}(\kappa-\alpha_1)^{(\ell)}(\kappa-\alpha_2)^{(\ell)}...(\kappa-\alpha_n)^{(\ell)}}{(\kappa-\beta_1)^{(\ell)}(\kappa-\beta_2)^{(\ell)}...(\kappa-\beta_n)^{(\ell)}}v(\kappa)$,

has a solution $v(\kappa) = \dfrac{c_1c^{\kappa}\Gamma(\kappa-\alpha_1-\ell+1)\Gamma(\kappa-\alpha_2-\ell+1)...\Gamma(\kappa-\alpha_n-\ell+1)}{\Gamma(\kappa-\beta_1-\ell+1)\Gamma(\kappa-\beta_2-\ell+1)...\Gamma(\kappa-\beta_n-\ell+1)}$.

Theorem 2.4.2. *If f is a non- linear function, then the generalized Clairaut's difference equation*

$$\ell u(\kappa) = \kappa \Delta_\ell u(\kappa) + f(\Delta_\ell u(\kappa)), \kappa \in \mathbb{N} \tag{2.26}$$

has a solution

$$u(\kappa) = \frac{1}{\ell}\{\kappa c + f(c)\} \tag{2.27}$$

and

$$(\kappa + \ell) + \frac{f(v(\kappa)) + \Delta_\ell v(\kappa) - f(v(\kappa))}{\Delta_\ell v(\kappa)} = 0 \tag{2.28}$$

yields another solution to (2.26) where $v(\kappa) = \Delta_\ell u(\kappa)$.

Proof. By taking $v(\kappa) = \Delta_\ell u(\kappa)$, (2.26) becomes

$$\ell u(\kappa) = \kappa v(\kappa) + f(v(\kappa)) \tag{2.29}$$

and yields

$$\ell v(\kappa) = (\kappa + \ell)v(\kappa + \ell) - \kappa v(\kappa) + f(v(\kappa + \ell)) - f(v(\kappa))$$

which is the same as

$$(\kappa + \ell)\Delta_\ell v(\kappa) + f(v(\kappa) + \Delta_\ell v(\kappa)) - f(v(\kappa)) = 0. \tag{2.30}$$

which yield (2.27) and (2.28), (2.30) is possible if either $\Delta_\ell v(\kappa) = 0$ or (2.27) hold □

Example 2.4.3. *The generalized Clairaut's equation*

$$\ell u(\kappa) = \kappa \Delta_\ell u(\kappa) + [\Delta_\ell u(\kappa)]^2, \kappa \in \mathbb{N} \tag{2.31}$$

has solutions

$$u(\kappa) = \frac{1}{\ell}[\kappa c + c^2] \tag{2.32}$$

and

$$u(\kappa) = \frac{1}{\ell}\left[\sum_{n=0}^{\ell-1} c_n e^{i\pi(2n+1)\frac{\kappa}{\ell}} - \frac{\ell}{4}\right]^2 - \frac{\kappa^2}{4\ell}. \tag{2.33}$$

Proof. Now (2.32) follows from (2.31), (2.27) and (2.28) corresponding to (2.31) of Theorem 2.4.2 and $v(\kappa) = \Delta_\ell u(\kappa)$ yield

$$v(\kappa + \ell) + v(\kappa) + \kappa + \ell = 0, \tag{2.34}$$

which has the general solution

$$v(\kappa) = \sum_{n=0}^{\ell-1} c_n e^{i\pi(2n+1)\frac{\kappa}{\ell}} - \frac{\kappa}{2} - \frac{\ell}{4}. \tag{2.35}$$

Now (2.33) follows from (2.31), $\Delta_\ell u(\kappa) = v(\kappa)$ and (2.35). □

Example 2.4.4. *The generalized Clairaut's difference equation*

$$\ell u(\kappa) = \kappa \Delta_\ell u(\kappa) + \frac{1}{\Delta_\ell u(\kappa)}, \kappa \in \mathbb{N} \tag{2.36}$$

has a solution

$$u(\kappa) = \frac{1}{\ell}\left[\kappa c + \frac{1}{c}\right], \tag{2.37}$$

and another solution

$$u(\kappa\ell + j) = \begin{cases} a_i \sum_{r=0}^{\frac{\kappa}{2}-1} \frac{[(2r-1)\ell+j]_{2\ell}^{(r)}}{(2r\ell+j)_{2\ell}^{(r)}} + \frac{1}{a_j} \sum_{r=0}^{\frac{\kappa}{2}-1} \frac{(2r\ell+j)_{2\ell}^{(r)}}{[(2r+1)\ell+j]_{2\ell}^{(r)}} + c_j \text{ if } \kappa \text{ is even} \\ a_j \sum_{r=0}^{\frac{\kappa-1}{2}} \frac{[(2r-1)\ell+j]_{2\ell}^{(r)}}{(2r\ell+j)_{2\ell}^{(r)}} + \frac{1}{a_j} \sum_{r=0}^{\frac{\kappa-1}{2}-1} \frac{(2r\ell+j)_{2\ell}^{(r)}}{[(2r+1)\ell+j]_{2\ell}^{(r)}} + c_j \text{ if } \kappa \text{ is odd,} \end{cases} \tag{2.38}$$

where a_j's and c_j's are arbitrary constants and $j = 0, 1, 2, \ldots, \ell - 1$.

Proof. Now (2.37) follows from (2.36) and (2.27).

Taking $v(\kappa) = \Delta_\ell u(\kappa)$ and (2.28) corresponding to (2.36) yields

$$v(\kappa)v(\kappa+\ell) = \frac{1}{\kappa+\ell}. \tag{2.39}$$

From $\Delta_\ell v(\kappa) \neq 0$, we get $(\kappa+\ell) + \frac{1}{\Delta_\ell v(\kappa)}\left[\frac{1}{v(\kappa+\ell)} - \frac{1}{v(\kappa)}\right] = 0$,

Since $\kappa \in \mathbb{N}$ implies $\kappa = (2n+1)\ell + j$ or $2n\ell + j$ for some $n \in \mathbb{N}$ and

$j \in \{0, 1, 2, \ldots, \ell-1\}$, by (2.5), we obtain a solution of (2.39) as

$$v(2n\ell + j) = \frac{[(2n-1)\ell + j]_{2\ell}^{(n)} v(j)}{(2n\ell + j)_{2\ell}^{(n)}}, \tag{2.40}$$

and

$$v[(2n+1)\ell + j] = \frac{(2n\ell + j)_{2\ell}^{(n)}}{[(2n+1)\ell + j]_{2\ell}^{(n+1)} v(j)}. \tag{2.41}$$

Now, (2.38) follows from (2.9), $u(\kappa) = \Delta_\ell^{-1} v(\kappa)$, (2.40) and (2.41). □

Theorem 2.4.5. *If* $c_0 c_n \neq 0$*, then* $u(\kappa) = \frac{\Gamma\left(\frac{\kappa}{\ell}+\lambda\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)}$ *is a solution to the generalized Euler difference equation*

$$\sum_{j=0}^{n} c_j \left[\kappa + (j-1)\ell\right]_\ell^{(j)} \Delta_\ell^j u(\kappa) = 0, \kappa \in \mathbb{N}(1) \tag{2.42}$$

if and only if λ *is a root of the equation*

$$\sum_{j=0}^{n} c_j \ell^j (\lambda)^{(j)} = 0. \tag{2.43}$$

Proof. By seeking a solution of (2.42) in the form $u(\kappa) = \frac{\Gamma\left(\frac{\kappa}{\ell}+\lambda\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)}$ with $\frac{\kappa}{\ell} + \lambda$ different from negative integer, from $\Delta_\ell^j u(\kappa) = \frac{\lambda^{(j)}\Gamma\left(\frac{\kappa}{\ell}+\lambda\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)}$ and (2.43) it follows that

$$\sum_{j=0}^{n} c_j \left[\kappa + (j-1)\ell\right]_\ell^{(j)} (\lambda)^{(j)} \frac{\Gamma\left(\frac{\kappa}{\ell}+\lambda\right)}{\Gamma\left(\frac{\kappa}{\ell}+i\right)}. \tag{2.44}$$

Since

$$\left(\frac{\kappa}{\ell}+i-1\right)^{(i)} = \frac{[\kappa+(i-1)\ell]_\ell^{(i)}}{\ell^i} = \frac{\Gamma\left(\frac{\kappa}{\ell}+i\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)},$$

(2.44) becomes

$$\sum_{j=0}^{n} c_i \ell^j (\lambda)^{(j)} \frac{\Gamma\left(\frac{\kappa}{\ell}+\lambda\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)} = 0,$$

which completes the proof of the theorem. □

Corollary 2.4.6. $u(\kappa) = \frac{\Gamma\left(\frac{\kappa}{\ell}+\lambda\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)}$ *is a solution to the difference equation*

$$\sum_{j=0}^{n} c_j \ell^{n-j} [\kappa+(i-1)\ell]_\ell^{(j)} \Delta_\ell^{(j)} u(\kappa) = 0 \tag{2.45}$$

if and only if λ *is a root of the equation*

$$\sum_{j=0}^{n} c_j \lambda^{(j)} = 0. \tag{2.46}$$

Proof. The proof follows by replacing c_j by $c_j \ell^{n-j}$ in (2.42) and $\ell^n \dfrac{\Gamma\left(\frac{\kappa}{\ell}+\lambda\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)} \neq 0$. □

Corollary 2.4.7. *The function* $u(\kappa) = \frac{\Gamma\left(\frac{\kappa}{\ell}+\lambda\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)}$ *is a solution of the difference equation*

$$\sum_{j=0}^{n} \frac{c_j}{\ell^j} [\kappa+(j-1)\ell]_\ell^{(j)} \Delta_\ell^{(j)} u(\kappa) = 0 \tag{2.47}$$

if and only if λ *is a root of the equation (2.46).*

Proof. The proof follows by replacing c_j by $\dfrac{c_j}{\ell^j}$ in (2.42). □

Corollary 2.4.8. *If* $\lambda = \alpha + i\beta$ *is a complex root of equation (2.43), then*

$$u_1(\kappa) = \frac{1}{\mu\left(\frac{\kappa}{\ell}\right)} \int_0^\infty e^{-t} t^{\frac{\kappa}{\ell}+\alpha-1} \cos(\beta \ln t)\, dt$$

and

$$u_2(\kappa) = \frac{1}{\mu\left(\frac{\kappa}{\ell}\right)} \int_0^\infty e^{-t} t^{\frac{\kappa}{\ell}+\alpha-1} \sin\left(\beta \ln t\right) dt$$

are two independent solutions of (2.42), (2.45) and (2.47).

Proof. The proof follows by taking $u_1(\kappa)$ and $u_2(\kappa)$ as

$$u_1(\kappa) = \frac{\Gamma\left(\frac{\kappa}{\ell} + \alpha + \frac{3}{2}i\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)} \text{ and } u_2(\kappa) = \frac{\Gamma\left(\frac{\kappa}{\ell} + \alpha - \frac{3}{2}i\right)}{\Gamma\left(\frac{\kappa}{\ell}\right)}.$$

and Example 2.4.1. □

Chapter 3

Higher Order Delta Operator and its Sum

3.1 Introduction

The fractional sum of a function f (or ν^{th} order delta integration) is defined by

$$(\Delta_a^{-\nu}u)(\kappa) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{\kappa-\nu}\frac{\Gamma(\kappa-s)}{\Gamma(\kappa-s-(\nu-1))}u(s), \tag{3.1}$$

where $\nu > 0$, f is defined for $s = a\; mod(1)$ and $\Delta^{-\nu}f$ is defined for $\kappa = a+\nu\; mod(1)$. Most of the mathematicians in this field are aware of this definition in phase are (Summation form), but they are not aware of another phase of equation (3.1). In our research, we are taking care of the second phase called exact form (closed form

) of equation (3.1). Hence, we discuss equation (3.1) in a detailed manner.

The authors are considering equation (3.1) as a definition for all $\nu > 0$. But our finding shows that the equation need not be a definition, it can be considered as a theorem for all positive integer $\nu = n \in \mathbb{N}(1)$.

3.2 Finite Fractional Order Difference

Let J_ℓ be a subset of $\mathbb{R}$ satisfying the condition that $a \in J_\ell$ if and ony if $a \pm \ell \in J_\ell$. In the following theorem, we are going to show that equation (3.1) is a theorem but not a definition if ν is a positive integer n and $a = 0$.

First ,we give an example for proving the equation (3.1) and which will be used in the subsequent derivation.

Example 3.2.1. *Consider the function* $u(\kappa) = \kappa^{(n)},\ n \geq 1,\ \kappa \in \mathbb{Z}$, *where* $\kappa^{(n)} = \kappa(\kappa-1)\cdots(\kappa-(n-1))$ *is the falling factorials.*

Now, $f(0) = 0$ *and* $\Delta\kappa^{(n)} = n\kappa^{(n-1)},\ \Delta^2\kappa^{(n)} = n^{(2)}\kappa^{(n-2)},\ \Delta^3\kappa^{(n)} = n^{(3)}\kappa^{(n-3)}, \cdots$ *and in general, we have*

$$\Delta^r\kappa^{(n)} = n^{(r)}\kappa^{(n-r)},\ r \leq n,\ \kappa^{(0)} = 1, \tag{3.2}$$

where $n^{(r)} = n(n-1)\cdots(n-(r-1))$. *From* (3.30), *and the extension of* (3.15), *we get*

$$\Delta^{-r}\kappa^{(n-r)} = \frac{\kappa^{(n)}}{n^{(r)}}. \tag{3.3}$$

If we replace n by n + r in (3.2), we get

$$\Delta^{-r}\kappa^{(n)} = \frac{\kappa^{(n+r)}}{(n+r)^{(r)}},\ n \geq 0. \tag{3.4}$$

It is clear that $\Delta^{-r}u(\kappa)\Big|_{\kappa=0} = \frac{\kappa^{(n+r)}}{(n+r)^{(r)}}\Big|_{\kappa=0} = 0,\ r = 0,1,2,3,\cdots$ *When* $r = 2$,

$$\Delta^{-2}\kappa^{(n)} = \frac{\kappa^{(n+2)}}{(n+2)^{(2)}}. \tag{3.5}$$

Substituting (3.5) in (3.29), we get

$$\frac{\kappa^{(n+2)}}{(n+2)^{(2)}} - \frac{0^{(n+2)}}{(n+2)^{(2)}} = \sum_{s=0}^{\kappa-2} \frac{\Gamma(\kappa-s)}{\Gamma(\kappa-s-1)} s^{(n)}, \tag{3.6}$$

$$\frac{\kappa^{(n+2)}}{(n+2)^{(2)}} = (\kappa-2)1^{(n)} + (\kappa-3)2^{(n)} + \cdots + 1(\kappa-2)^{(n)}.$$

Suppose that $\kappa = 1000$, *(3.6) becomes*

$$\frac{1000^{(n+2)}}{(n+2)^{2}} = (998)1^{(n)} + (997)2^{(n)} + \cdots + 1(998)^{(n)}.$$

Now, since $\Gamma(n) = (n-1)\Gamma(n-1)$, *(3.6) can be expressed as*

$$\frac{\kappa^{(n+2)}}{(n+2)^{(2)}} = (\kappa-1)0^{(n)} + (\kappa-2)1^{(n)} + (\kappa-3)2^{(n)} + \cdots + 1(\kappa-2)^{(n)}. \tag{3.7}$$

Special cases: when $n = 0$, $0! = 1$, we have

$$\frac{\kappa^{(2)}}{2^{(2)}} = 1 + 2 + 3 + \cdots + (\kappa-3) + (\kappa-2) + (\kappa-1), \tag{3.8}$$

which is a well known formula.

For verification, if we take $n = 1$ and $\kappa = 5$ in (3.7), we can have

$$\frac{5^{(3)}}{3^{(2)}} = (4)0^{(1)} + (3)1^{(1)} + (2)2^{(1)} + (1)3^{(1)},$$

(i.e) $\dfrac{5.4.3}{3.2} = 0 + 3 + 2(2) + 1(3) = 10.$

Similarly one can verify formula (3.7) for any positive integer n and κ.

Example 3.2.2. *By taking $r = 1$ in (3.4), we get*

$$\Delta^{-1}\kappa^{(n)} = \frac{\kappa^{(n+1)}}{(n+1)}. \tag{3.9}$$

If we take $u(\kappa) = \kappa^{(n)}$, $\kappa \in \mathbb{N}(1)$ and applying (3.9) in (3.24), we arrive

$$\frac{\kappa^{(n+1)}}{(n+1)} - \frac{0^{(n+1)}}{(n+1)} = 0^{(n)} + 1^{(n)} + 2^{(n)} + \cdots + (\kappa - 1)^{(n)}. \tag{3.10}$$

By replacing κ by $\kappa + 1$ in (3.10), it is obvious that

$$\frac{(\kappa+1)^{(n+1)}}{(n+1)} = 1^{(n)} + 2^{(n)} + \cdots + \kappa^{(n)}. \tag{3.11}$$

For verification, if we take $\kappa = 5$ and $n = 3$, then (3.11) becomes
$\dfrac{6^{(4)}}{4} = 1^{(3)} + 2^{(3)} + 3^{(3)} + 4^{(3)} + 5^{(3)}$, *then* $\dfrac{6.5.4.3}{4} = 0 + 0 + 3.2.1 + 4.3.2 + 5.4.3.$
(i.e), $\dfrac{6.5.4.3}{4} = 6 + 24 + 60 = 90.$ *We will use the formula (3.11) in the main derivation.*

The general form of (3.10) is given by

$$\frac{(\kappa+1)^{(n+1)}}{(n+1)} - \frac{a^{(n+1)}}{(n+1)} = a^{(n)} + (a+1)^{(n)} + \cdots + (\kappa-1)^{(n)}, \ \forall\ \kappa \in \mathbb{R} \text{ if } \kappa - a \in \mathbb{N}(1). \tag{3.12}$$

For verification, if we take $\kappa = 4.5$, $a = 1.5$ and $n = 3$ in (3.12), we have

$$\frac{(4.5)^{(4)}}{4} - \frac{(1.5)^{(4)}}{4} = (1.5)^{(3)} + (2.5)^{(3)} + (3.5)^{(3)}$$

$$\frac{(4.5)(3.5)(2.5)(1.5)}{4} - \frac{(1.5)(0.5)(-0.5)(-1.5)}{4} = (1.5)(0.5)(-0.5)$$

$+ (2.5)(1.5)(0.5) + (3.5)(2.5)(1.5)$

$$\frac{1}{4}\Big(\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2} - \frac{3}{2}.\frac{1}{2}.\frac{1}{2}.\frac{3}{2}\Big) = -\frac{3}{2}.\frac{1}{2}.\frac{1}{2} + \frac{5}{2}.\frac{3}{2}.\frac{1}{2} + \frac{7}{2}.\frac{5}{2}.\frac{3}{2}$$

$$\frac{9.7.5.3}{8} - \frac{3.1.3}{8} = -3.1.1 + 5.3.1 + 7.5.3 = -1 + 5 + 35 = 37$$

Thus, $9.7.5 - 3 = 8(39) = 312.$

Hence, (3.12) is valid for all real κ *and* a *such that* $\kappa - a \in \mathbb{N}(1)$.

Theorem 3.2.3. *Assume that* J_1 *be a subset of real numbers such that* $0 \in J_1$ *and* $\kappa \in J_1$ *implies* $\kappa \pm 1 \in J_1$ *and* $u : J_1 \to \mathbb{R}$ *be a real valued function. If* $\Delta^{-r}u(\kappa)\Big|_{\kappa=0} = 0$ *for* $r = 1, 2, 3, \cdots n,$ *then*

$$(\Delta^{-n}u)(\kappa) = \frac{1}{(n-1)!}\sum_{s=0}^{\kappa-n}(t-s-1)^{(n-1)}u(s), \tag{3.13}$$

where $\kappa \in J_1,\ \kappa \geq n+1$ *and integer .*

Proof: *From the definition of the delta operator* Δ *on* $u(\kappa)$*, we have*

$$\Delta u(\kappa) = u(\kappa+1) - u(\kappa),\ \ \kappa \in J_1 \tag{3.14}$$

and if $\Delta v(\kappa) = u(\kappa),\ \kappa \in J_1,$ *then*

$$v(\kappa) = \Delta^{-1}u(\kappa). \tag{3.15}$$

Equation (3.14) and (3.15) are basic definitions. Remember that in (3.15), $v(\kappa)$ *is an delta inverse of* $u(\kappa)$*. Here, we have assumed that*

$$\Delta^{r}u(\kappa)\Big|_{\kappa=0} = 0,\ \ r = 0, 1, 2, \cdots, n. \tag{3.16}$$

By taking

$$\Delta^{-1}u(\kappa) = v(\kappa)(say), \tag{3.17}$$

from (3.14) *and* (3.15), *we write*

$$u(\kappa) = \Delta v(\kappa) = v(\kappa+1) - v(\kappa),\ \ \kappa \in J_1.$$

Replacing κ *by* $\kappa - 1$, *we get* $u(\kappa-1) = v(\kappa) - v(\kappa-1)$, *which gives*

$$v(\kappa) = u(\kappa-1) + v(\kappa-1). \tag{3.18}$$

Replacing κ *by* $\kappa - 1$ *in* (3.18), *we get*

$$v(\kappa-1) = u(\kappa-2) + v(\kappa-2). \tag{3.19}$$

Substituting (3.19) *in* (3.18), *we get*

$$v(\kappa) = u(\kappa-1) + u(\kappa-2) + v(\kappa-2). \tag{3.20}$$

Replacing κ *by* $\kappa-2, \kappa-3, \cdots \kappa-(m-1)$ *respectively in* (3.18), *we will be obtaining* $v(\kappa-2), v(\kappa-3), \cdots, v(\kappa-(m-1))$. *Substituting all these values in* (3.20), *we find*

$$v(\kappa) = u(\kappa-1) + u(\kappa-2) + \cdots + u(\kappa-m) + v(\kappa-m),$$

which can be expressed, using (3.17), *as*

$$\Delta^{-1}u(\kappa) - \Delta^{-1}u(\kappa-m) = u(\kappa-1) + u(\kappa-2) + \cdots + u(\kappa-m). \tag{3.21}$$

If we take $\kappa - m = a$ *in* (3.21), *then we can arrange* (3.21) *as*

$$\Delta^{-1}u(\kappa) - \Delta^{-1}u(a) = \sum_{s=a}^{\kappa-1} u(s), \;\; \kappa \in \mathbb{Z}, \;\; \kappa \geq a+1 \tag{3.22}$$

or

$$\Delta_a^{-1}u(\kappa) = \frac{1}{0!}\sum_{s=a}^{\kappa-1}(t-s-1)^{(0)}u(s), \;\; \kappa \in J_1, \;\; \kappa - 1 - a \in \mathbb{N}(1). \tag{3.23}$$

When $m = \kappa$ *and* $a = 0$, *relation* (3.23) *becomes*

$$\Delta_0^{-1}u(\kappa) = \sum_{s=0}^{\kappa-1}(t-s-1)^{(0)}u(s), \;\; \kappa \in \mathbb{Z}, \;\; \kappa \geq 1, \tag{3.24}$$

where $\Delta_0^{-1}u(\kappa) = \Delta^{-1}u(\kappa) - \Delta^{-1}u(\kappa)\Big|_{\kappa=0}$.

By assuming $\Delta^{-1}u(\kappa)\Big|_{\kappa=0}$, (3.24) *becomes*

$$\Delta^{-1}u(\kappa) = u(0) + u(1) + u(2) + \cdots + u(\kappa-1). \tag{3.25}$$

If we replace κ *by* $\kappa - 1$ *in* (3.25) *and* $\kappa \geq 0$, *we have*

$$\Delta^{-1}u(\kappa-1) = u(0) + u(1) + u(2) + \cdots + u(\kappa-2).$$

Similarly,

$\Delta^{-1}u(\kappa-2) = u(0) + u(1) + u(2) + \cdots + u(\kappa-3).$

$\Delta^{-1}u(\kappa-3) = u(0) + u(1) + u(2) + \cdots + u(\kappa-4).$

$\vdots$

$\Delta^{-1}u(\kappa-(\kappa-1)) = u(0)$

$\Delta^{-1}u(\kappa-\kappa) = \Delta^{-1}u(0) = 0$ *(assumption)*

Adding the above all expressions, we get

$\Delta^{-1}u(\kappa-1) + \Delta^{-1}u(\kappa-2) + \cdots + \Delta^{-1}u(1) + \Delta^{-1}u(0)$

$$= 1u(\kappa-2)+2u(\kappa-3)+3u(\kappa-4)+\cdots+(\kappa-1)u(0). \quad (3.26)$$

If we replace f by $\Delta^{-1}f$ in (3.25), we will be getting

$$\Delta^{-1}(\Delta^{-1}u(\kappa)) = \Delta^{-1}u(0) + \Delta^{-1}u(1) + \Delta^{-1}u(2) + \cdots + \Delta^{-1}u(\kappa-1) \quad (3.27)$$

and (3.26) becomes

$$\Delta^{-2}u(\kappa) = (\kappa-1)u(0) + (\kappa-2)u(1) + \cdots + 2u(\kappa-3) + 1u(\kappa-2), \quad (3.28)$$

where $\Delta^{-2}u(\kappa) = \Delta^{-1}(\Delta^{-1}u(\kappa)),\ \kappa \in \mathbb{Z},\ \kappa \geq 2.$

Now, (3.28) can be expressed as

$$\Delta^{-2}u(\kappa) - \Delta^{-2}u(0) = \frac{1}{1!}\sum_{s=0}^{\kappa-2}(t-s-1)^{(1)}u(s),\ \kappa \geq 2,\ \kappa \in \mathbb{Z} \quad (3.29)$$

which is the same as, Since $\overset{-2}{\Delta} u(0) = 0$,

$$\Delta_0^{-2}u(\kappa) = \frac{1}{1!}\sum_{s=0}^{\kappa-2}(t-s-1)^{(1)}u(s),\ \kappa \in \mathbb{N}(2). \quad (3.30)$$

Considering the equation (3.28), we arrive

$$\Delta^{-2}u(\kappa) = (\kappa-1)u(0) + (\kappa-2)u(1) + (\kappa-3)u(2) + \cdots + 2u(\kappa-3) + 1u(\kappa-2).$$

Replacing κ by $\kappa-1$ in (3.28) gives

$$\Delta^{-2}u(\kappa-1) = (\kappa-2)u(0)+(\kappa-3)u(1)+(\kappa-4)u(2)+\cdots+2u(\kappa-4)+1u(\kappa-3). \quad (3.31)$$

Again, replacing κ by $\kappa-2$ in (3.28) yields

$$\Delta^{-2}u(\kappa-2) = (\kappa-3)u(0)+(\kappa-4)u(1)+(\kappa-5)u(2)+\cdots+2u(\kappa-5)+1u(\kappa-4). \quad (3.32)$$

By replacing κ by $\kappa-3$ in (3.32), we get

$$\Delta^{-2}u(\kappa-3)=(\kappa-4)u(0)+(\kappa-5)u(1)+(\kappa-6)u(2)+\cdots+2u(\kappa-6)+1u(\kappa-5).$$

By continuing this procedure, we get

$\Delta^{-2}u(\kappa-(\kappa-1))=1.u(0).$

$\Delta^{-2}u(\kappa-\kappa)=\Delta^{-2}u(0)=0.$

Adding all the above expressions starting from (3.31), we can arrange

$$\begin{aligned}\Delta^{-2}u(\kappa-1)+\Delta^{-2}u(\kappa-2)+\Delta^{-2}u(\kappa-3)+\cdots+\Delta^{-2}u(1)+\Delta^{-2}u(0)\\ =[1+2+\cdots+(\kappa-2)]u(0)+[1+2+\cdots+(\kappa-3)]u(1)\\ +[1+2+\cdots+(\kappa-4)]u(2)+\cdots+[1+2]u(\kappa-4)+1[u(\kappa-3)].\end{aligned}\tag{3.33}$$

Replacing f by $\Delta^{-2}f$ in (3.25), we get

$$\Delta^{-2}u(\kappa-1)+\Delta^{-2}u(\kappa-2)+\Delta^{-2}u(\kappa-3)+\cdots+\Delta^{-2}u(1)+\Delta^{-2}u(0)=\Delta^{-3}u(\kappa).\tag{3.34}$$

Applying (3.8) and (3.34) in (3.33) gives

$$\begin{aligned}\Delta^{-3}u(\kappa)&=\frac{(\kappa-1)^{(2)}}{2}u(0)+\frac{(\kappa-2)^{(2)}}{2}u(1)+\cdots+\frac{3^{(2)}}{2}u(\kappa-4)+\frac{2^{(2)}}{2}u(\kappa-3)\\ &=\frac{1}{2!}\Big((\kappa-1)(\kappa-2)u(0)+(\kappa-2)(\kappa-3)u(1)+(\kappa-3)(\kappa-4)u(2)+\cdots\\ &\qquad+3.2u(\kappa-4)+2.1u(\kappa-3)\Big).\\ &=\frac{1}{2!}\Big((\kappa-1)(\kappa-2)u(0)+(\kappa-2)(\kappa-3)u(1)+\cdots\\ &\qquad+3.2u(\kappa-4)+2.1u(\kappa-3)\Big).\end{aligned}$$

$$= \frac{1}{2!}\Big((\kappa-1)^{(2)}u(0) + (\kappa-2)^{(2)}u(1) + (\kappa-3)^{(3)}u(2) + \cdots + 3.2u(\kappa-4) + 2.1u(\kappa-3)\Big).$$

$$\Delta^{-3}u(\kappa) = \frac{1}{2!}\sum_{s=0}^{\kappa-3}(t-s-1)^{(2)}u(s). \tag{3.35}$$

We prove the result (3.2.3) by induction on n.

Hypothesis: Assume that the following relation holds;

$$\Delta^{-(n-1)}u(\kappa) = \frac{1}{\Gamma(n-1)}\sum_{s=0}^{\kappa-(n-1)}(t-s-1)^{(n-2)}u(s). \tag{3.36}$$

where the falling factorial,

$$(\kappa-s-1)^{(n-2)} = (\kappa-s-1)(\kappa-s-2)(\kappa-s-3)\cdots(\kappa-s-(n-2)), \tag{3.37}$$

Substituting (3.37) in (3.36), we obtain

$$\Delta^{-(n-1)}u(\kappa) = \frac{1}{(n-2)!}\sum_{s=0}^{\kappa-(n-1)}(\kappa-s-1)^{(n-2)}u(s). \tag{3.38}$$

By expanding the series (3.38), we derive

$$\Delta^{-(n-1)}u(\kappa) = \frac{1}{(n-2)!}\Big((\kappa-1)^{(n-2)}u(0) + (\kappa-2)^{(n-2)}u(1) + (\kappa-3)^{(n-2)}u(2) + \cdots + (n-1)^{(n-2)}u(\kappa-n) + (n-2)^{(n-2)}u(\kappa-(n-1))\Big). \tag{3.39}$$

Replacing κ by $\kappa-1$ in (3.40) yields

$$\Delta^{-(n-1)}u(\kappa-1) = \frac{1}{(n-2)!}\Big((\kappa-2)^{(n-2)}u(0)+(\kappa-3)^{(n-2)}u(1)+(\kappa-4)^{(n-2)}u(2) (\kappa-5)^{(n-2)}u(3) + \cdots + (n-2)^{(n-2)}u(\kappa-n)\Big). \tag{3.40}$$

$$\Delta^{-(n-1)}u(\kappa-2) = \frac{1}{(n-2)!}\Big((\kappa-3)^{(n-2)}u(0)+(\kappa-4)^{(n-2)}u(1)+(\kappa-5)^{(n-2)}u(2)$$

$$(\kappa-6)^{(n-2)}u(3)+\cdots+(n-2)^{(n-2)}u(\kappa-n-1)\Big).$$

$$\Delta^{-(n-1)}u(\kappa-3) = \frac{1}{(n-2)!}\Big((\kappa-4)^{(n-2)}u(0)+(\kappa-5)^{(n-2)}u(1)+(\kappa-6)^{(n-2)}u(2)$$

$$(\kappa-7)^{(n-2)}u(3)+\cdots+(n-2)^{(n-2)}u(\kappa-n-2)\Big).$$

$$\vdots$$

$$\Delta^{(n-1)}u(\kappa-(\kappa-1)) = \frac{(n-2)^{(n-2)}}{(n-2)!}u(0).$$

$\Delta^{(n-1)}u(0) = 0$ (by assumption)

By adding all the above equation starting from (3.40), we find

$$\Delta^{-(n-1)}u(\kappa-1)+\Delta^{-(n-1)}u(\kappa-2)+\Delta^{-(n-1)}u(\kappa-3)+\cdots+\Delta^{-(n-1)}u(0)$$

$$= \frac{1}{(n-2)!}\Big([(\kappa-2)^{(n-2)}+(\kappa-3)^{(n-2)}+\cdots+(n-2)^{(n-2)}]u(0)$$

$$+[(\kappa-3)^{(n-2)}+(\kappa-4)^{(n-2)}+\cdots+(n-2)^{(n-2)}]u(1)$$

$$+[(\kappa-4)^{(n-2)}+(\kappa-5)^{(n-2)}+\cdots+(n-2)^{(n-2)}]u(2)$$

$$+\cdots+(n-2)^{(n-2)}u(\kappa-n)\Big). \tag{3.41}$$

Replacing u by $\Delta^{-1}u$ in (3.25), we get

$$\Delta^{-1}\Delta^{-(n-1)}u(\kappa) = \Delta^{-(n-1)}u(\kappa-1)+\Delta^{-(n-1)}u(\kappa-2)+\cdots+\Delta^{-(n-1)}u(0),$$

which is same as

$$\Delta^{-n}u(\kappa) = \Delta^{-(n-1)}u(\kappa-1)+\Delta^{-(n-1)}u(\kappa-2)+\cdots+\Delta^{-(n-1)}u(0). \tag{3.42}$$

Replacing κ by $\kappa-1$, n by $n-2$ and a by $n-2$ in (3.12), we obtain

$$\frac{(\kappa-1)^{(n-1)}}{(n-1)}-\frac{(n-2)^{(n-1)}}{(n-1)}=(n-2)^{(n-2)}+(n-1)^{(n-2)}+\cdots+(\kappa-2)^{(n-2)}, \quad (3.43)$$

Since $(n-2)^{(n-1)}=0$, we can arrive at

$$\frac{(\kappa-1)^{(n-1)}}{(n-1)}=(\kappa-2)^{(n-2)}+(\kappa-3)^{(n-2)}+\cdots+(n-1)^{(n-2)}+(n-2)^{(n-2)}. \quad (3.44)$$

Similarly, it is easy to obtain

$$\frac{(\kappa-2)^{(n-1)}}{(n-1)}=(\kappa-3)^{(n-2)}+(\kappa-4)^{(n-2)}+\cdots+(n-1)^{(n-2)}+(n-2)^{(n-2)} \quad (3.45)$$

$$\frac{(\kappa-3)^{(n-1)}}{(n-1)}=(\kappa-4)^{(n-2)}+(\kappa-5)^{(n-2)}+\cdots+(n-1)^{(n-2)}+(n-2)^{(n-2)}$$

and so on, and finally we get

$$\frac{(n-1)^{(n-1)}}{(n-1)}=(n-2)^{(n-2)}. \quad (3.46)$$

Substituting (3.42)-(3.46) in (3.41), we obtain

$$\Delta^{-n}u(\kappa)=\frac{1}{(n-2)!}\Big(\frac{(\kappa-1)^{(n-1)}}{(n-1)}u(0)+\frac{(\kappa-2)^{(n-1)}}{(n-1)}u(1)+\cdots+\frac{(n-1)^{(n-1)}}{(n-1)}u(\kappa-n)\Big),$$

which is same as

$$\Delta^{-n}u(\kappa)-\Delta^{-n}u(0)=\frac{1}{(n-2)!}\Big((\kappa-1)^{(n-1)}u(0)+(\kappa-2)^{(n-1)}u(1)$$
$$+(\kappa-3)^{(n-1)}u(2)+\cdots+(n-1)^{(n-1)}u(\kappa-n)\Big). \quad (3.47)$$

as $\Delta^{-n}u(0)=0$.

Now, from the property of falling factorial, we find

$$(\kappa-1)^{(n-1)}=(\kappa-1)(\kappa-2)(\kappa-3)\cdots(\kappa-(n-1)),$$

and (3.47) becomes, since $\overset{-n}{\Delta} u(0) = 0$

$$\Delta_0^{-n} u(\kappa) = \frac{1}{(n-1)!}\Big((\kappa-1)^{(n-1)}u(0)+(\kappa-2)^{(n-2)}u(1)+\cdots+(\kappa-(n-1))^{\kappa-(n-1)}u(\kappa-n)\Big),$$

which gives ,

$$\Delta_0^{-n} u(\kappa) = \frac{1}{(n-1)!}\sum_{s=0}^{k-n}(\kappa-s-1)^{(n-1)}u(s),\ \ 1 \le n \le \kappa \in \mathbb{N}(1). \tag{3.48}$$

and the proof ins complete.

Corollary 3.2.4. *Let* $\kappa \in \mathbb{R} = J_1$ *and* $\kappa - n - a \in \mathbb{N}(1)$*. If* $\Delta^{-r}u(\kappa)\Big|_{\kappa=a} = 0$ *for* $r = 0, 1, 2, \cdots, n,$ *then*

$$\Delta_a^{-n} u(\kappa) = \frac{1}{(n-1)}\sum_{s=a}^{\kappa-n}(\kappa-s-1)^{(n-1)}u(s), \tag{3.49}$$

where $\Delta_a^{-n}u(\kappa) = \Delta^{-n}u(\kappa) - \Delta^{-n}u(a).$

Proof: *The proof follows by taking* $\kappa - m = a$ *in the previous derivation.*

Example 3.2.5. *Consider the function* $u(\kappa) = (\kappa-5)^{(3)},$

$k \in \mathbb{N}(3) = \{3, 4, \cdots\}, u(5) = 0, \quad \Delta^{-1}u(\kappa) = \dfrac{(\kappa-5)^{(4)}}{4}$ *and* $\overset{-1}{\Delta} u(\kappa)\Big|_{\kappa=s} = 0.$

$$\begin{aligned}\Delta(\kappa-5)^{(3)} &= (\kappa-4)^{(3)} - (\kappa-5)^{(3)}\\ &= (\kappa-4)(\kappa-5)(\kappa-6) - (\kappa-5)(\kappa-6)(\kappa-7)\\ &= (\kappa-5)(\kappa-6)[\kappa-4-\kappa+7] = (\kappa-5)(\kappa-6).3\\ &= 3.(\kappa-5)^{(2)}\end{aligned}$$

Similarly, $\Delta(\kappa-5)^{(4)} = 4.(\kappa-5)^{(3)}$ *yields that*

$\Delta^{-1}(\kappa-5)^{(3)} = \dfrac{(\kappa-5)^{(4)}}{4}$ *and* $\Delta^{-1}(\kappa-5)^{(3)}\Big|_{\kappa=5} = 0$

$$\Delta^{-2}(\kappa-5)^{(3)} = \frac{(\kappa-5)^{(5)}}{5.4} \text{ and } \Delta^{-2}(\kappa-5)^{(3)}\Big|_{\kappa=5} = 0$$
$$\Delta^{-3}(\kappa-5)^{(3)} = \frac{(\kappa-5)^{(4)}}{6.5.4} \text{ and } \Delta^{-3}(\kappa-5)^{(3)}\Big|_{\kappa=5} = 0$$

and so on.

Taking $n = 3,\ \kappa = 11,\ a = 5$ *in* (3.49), *we get*

$$\Delta_5^{-3}(\kappa-5)^{(3)} = \frac{1}{2!}\sum_{s=5}^{8}\frac{(\kappa-s-1)!}{(\kappa-s-3)!}(s-5)^{(3)}, \tag{3.50}$$

$$\frac{(\kappa-5)^{(6)}}{6.5.4} - \frac{0^{(6)}}{6.5.4} = \frac{1}{2!}\sum_{s=5}^{8}(\kappa-s-1)(\kappa-s-2)(s-5)^{(3)}$$

Putting $\kappa = 11$ *gives*

$$\frac{6^{(6)}}{6.5.4} = \frac{1}{2!}\sum_{s=5}^{8}(10-s)(9-s)(s-5)^{(3)} = \frac{6.5.4.3.2.1}{6.5.4} = (3)(2)(1),$$

which completes the verification of (3.48).

The following example shows that the above procedure is not applicable for geometric and other functions.

Example 3.2.6. *Consider the function* $u(\kappa) = 2^\kappa,\ 2^0 \neq 0,$

Now $\Delta 2^\kappa = 2^{\kappa+1} - 2^\kappa = (2-1)2^\kappa = 2^\kappa.$

Similarly, $\Delta^2 2^\kappa = 2^\kappa$ *gives* $\Delta^{-2}2^\kappa = 2^\kappa$, $2^0 \neq 0$ *and*

$$\Delta^3 2^\kappa = 2^\kappa \text{ implies } \Delta^{-3}2^\kappa = 2^\kappa.$$

For $n = 3$, (3.48) *can be expressed as*

$$\Delta_0^{-3}u(\kappa) = \frac{1}{2!}\sum_{s=0}^{\kappa-3}(\kappa-s-1)(\kappa-s-2)u(s).$$

If $u(\kappa) = 2^\kappa$, then we have

$$\Delta^{-3}2^\kappa - \Delta^{-3}2^\kappa\Big|_{\kappa=0} = \frac{1}{2!}\sum_{s=0}^{\kappa-3}(\kappa - s - 1)(\kappa - s - 2)2^s.$$

Taking $\kappa = 5$, we get

$$2^5 - 2^0 = \frac{1}{2!}\sum_{s=0}^{2}(5 - s - 1)(5 - s - 2)2^s = \frac{1}{2!}[4.3.2^0 + 3.2.2 + 2.1.2^2],$$

which is not true. This example shows that $\Delta^{-1}2^\kappa = \Delta^{-2}2^\kappa = \Delta^{-3}2^\kappa = 2^\kappa$ is not true.

Suppose that equation (3.1) is true for all real ν, it should be true for positive integer. The correponding example 3.2.6 is not true for the function u if $\Delta^{-r}u(\kappa)\Big|_{\kappa=a} = 0$ for all $r = 0, 1, 2, \cdots, m (m = \nu)$.

Hence, a closed form for the equation (3.1) has to be arrived atleast for $\nu = m$(positive integer). In the next theorem we present such betterclosed form relation.

Theorem 3.2.7. *Let $f : \mathbb{N}(0) \to \mathbb{R}$ be a function. For $\kappa \in \mathbb{N}(1)$, if $\Delta^r v(\kappa) = u(\kappa), \Delta^{-r}u(\kappa)\Big|_{\kappa=0} = A_r$, $r = 0, 1, 2, \cdots, n$ then we take $v(\kappa) = \Delta^{-r}u(\kappa)$ and Also assume that $G^n_n(\kappa) = \Delta^{-n}u(\kappa) - A_n - A_{n-1}\frac{\kappa^{(1)}}{\Gamma(2)} - A_{n-2}\frac{\kappa^{(2)}}{\Gamma(3)} - \cdots - A_1\frac{\kappa^{(n-1)}}{\Gamma(n)}$. Then, we have a closed form for (3.1) as*

$$G^n_0(\kappa) - G^n_0(0) = \frac{1}{(n-1)!}\sum_{s=0}^{\kappa-n}(\kappa - s - 1)^{(n-1)}u(s), \;\; \kappa - n \geq 1. \tag{3.51}$$

Proof: *From the proof of Theorem 3.2.3, the equation (3.21) is expressed as*

$$\Delta^{-1}u(\kappa) - \Delta^{-1}u(\kappa - m) = u(\kappa - 1) + u(\kappa - 2) + u(\kappa - 3) + \cdots + u(\kappa - m).$$

Assume that $\kappa - m = a$ *and* $u(\kappa - m)$ *is fixed.*

Now, by the new notation, the above relation becomes

$$\Delta^{-1}u(\kappa) - A_1 = u(\kappa - 1) + u(\kappa - 2) + u(\kappa - 3) + u(\kappa - 4) + \cdots + u(a) \quad (3.52)$$

where $A_1 = \Delta^{-1}u(\kappa)\Big|_{\kappa=a}$, *is considered as constant.*

If we replace κ *by* $\kappa - 1, \kappa - 2, \kappa - 3, \cdots$ *and a=0 in* (3.52), *we can get*

$\Delta^{-1}u(\kappa - 1) - A_1 = u(\kappa - 2) + u(\kappa - 3) + u(\kappa - 4) + \cdots + u(0)$

$\Delta^{-1}u(\kappa - 2) - A_1 = u(\kappa - 3) + u(\kappa - 4) + u(\kappa - 5) + \cdots + u(0)$

$\Delta^{-1}u(\kappa - 3) - A_1 = u(\kappa - 4) + u(\kappa - 5) + u(\kappa - 6) + \cdots + u(0)$

$\Delta^{-1}u(\kappa - 4) - A_1 = u(\kappa - 5) + \cdots + u(0)$

and so on $\Delta^{-1}u(2) - A_1 = u(1) + u(0)$

$\Delta^{-1}u(1) - A_1 = u(0)$

$\Delta^{-1}u(0) - A_1 = 0$

Adding above expression, we find (as in the proof of Theorem 3.2.3)

$$\Delta^{-2}u(\kappa) - \Delta^{-2}u(0) - A_1\kappa = (\kappa - 1)u(0) + (\kappa - 2)u(1) + \cdots + 2u(\kappa - 3) + 1u(\kappa - 4),$$

which is the same as

$$\Delta^{-2}u(\kappa) - A_2 - A_1\kappa = (\kappa - 1)u(0) + (\kappa - 2)u(1) + \cdots + 2u(\kappa - 3) + 1u(\kappa - 4), \quad (3.53)$$

where $A_2 = \Delta^{-2}u(\kappa)\Big|_{\kappa=0}$ *is considered as constant*

If we replace κ *by* $\kappa - 1, \kappa - 2, \kappa - 3, \cdots$ *and* $\kappa - \kappa$ *in* (3.53) *and adding the* *corresponding expressions, we can easily find that*

$$\overset{-3}{\Delta} u(\kappa) - A_3 - A_2\kappa - A_1\frac{\kappa^{(2)}}{2} = \frac{1}{2!}\sum_{s=0}^{\kappa-3}(\kappa - s - 1)^{(2)}u(s),$$

where $A_3 = \Delta^{-3}u(\kappa)\Big|_{\kappa=0}$ *is constant*

By induction hypothesis, we assume that

$$\overset{-(n-1)}{\Delta} u(\kappa) - A_{n-1} - A_{n-2}\kappa - A_{n-3}\frac{\kappa^{(2)}}{2!} - A_{n-4}\frac{\kappa^{(3)}}{3!} - \cdots - A_1\frac{\kappa^{(n-2)}}{(n-2)!}$$

$$= \frac{1}{(n-2)!}\sum_{s=0}^{\kappa-(n-1)}(t - s - 1)^{(n-2)}u(s), \tag{3.54}$$

where $A_r = \Delta^{-1}u(\kappa)\Big|_{\kappa=0}$, $r = 1, 2, 3, \cdots, n-1$.

Replacing κ *by* $\kappa - 1, \kappa - 2, \cdots, \kappa - \kappa$ *in* (3.55) *and using the formula*

$$1^{(r)} + 2^{(r)} + \cdots + (\kappa - 1)^{(r)} = \frac{\kappa^{(r+1)}}{(r+1)}$$

and adding the corresponding expressions, it is easy to arrive

$$\Delta^{-n}u(\kappa) - A_n - A_{n-1}\frac{\kappa^{(1)}}{1!} - A_{n-2}\frac{\kappa^{(2)}}{2!} - \cdots - A_1\frac{\kappa^{(n-1)}}{(n-1)!}$$

$$= G_0^n(\kappa) - G_0^n(0) = \frac{1}{(n-1)!}\sum_{s=0}^{\kappa-n}(\kappa - s - 1)^{(n-1)}u(s), \tag{3.55}$$

where $A_r = \Delta^{-r}u(\kappa)\Big|_{\kappa=0}$ *is constant and the proof is complete.*

Corollary 3.2.8. *If* A_r^{15} *are zero for* $r = 1, 2, \cdots, n$, *then* (3.55) *becomes* (3.1).

Remark 3.2.9. *If we denote LHS term of* (3.51) *as* $G_0^n(\kappa) - G_0^n(0) = F_0^n(\kappa)$, *then* (3.55) *can be expressed by*

$$F_0^n(\kappa) = \frac{1}{(n-1)!}\sum_{s=0}^{\kappa-a}(\kappa - s - 1)^{(n-1)}u(s), \ \kappa - a - n \in \mathbb{N}(1). \tag{3.56}$$

Corollary 3.2.10. *Assume that* $\kappa - n - a \in \mathbb{N}(1)$. *Then, if we denote* $G_n(\kappa) = \Delta^{-n}u(\kappa) - A_n - A_{n-1}\frac{\kappa^{(1)}}{1!} - A_{n-2}\frac{\kappa^{(2)}}{2!} - \cdots - A_1\frac{\kappa^{(n-1)}}{(n-1)!}$. *Then we have*

$$G_n(\kappa) - G_n(a) = \frac{1}{(n-1)!}\sum_{s=a}^{\kappa-n}(\kappa - s - 1)^{(n-1)}u(s), \tag{3.57}$$

where $A_r = \Delta^{-1}u(\kappa)\Big|_{\kappa=0}$ *for* $r = 1, 2, 3, \cdots, n$.

Proof: *Replacing* κ *by* a *in* (3.53), *we obtain*

$$\Delta^{-n}u(a) - A_n - A_{n-1}\frac{a^{(1)}}{1!} - A_{n-2}\frac{a^{(2)}}{2!} - \cdots - A_1\frac{a^{(n-1)}}{(n-1)!}$$

$$= G_n(a) = \frac{1}{(n-1)!}\sum_{s=0}^{a-n}(a - s - 1)^{n-1}u(s). \tag{3.58}$$

Now (3.56) *follows by substracting* (3.58)*from*(3.55).

Corollary 3.2.11. *Let* $J_1 = \mathbb{R} - \mathbb{Z}$ *be the set of all real numbers not containing negative integers . Let* $u : J_1 \to \mathbb{R}$ *be a function. Let* n *be positive integer and choose* κ *and* a *such that* $\kappa - n - a \in \mathbb{N}(1)$. *Let* $B_r = \Delta^{-r}u(\kappa)\Big|_{\kappa=a}$ *and define* $G_a^n(\kappa) = \Delta^{-n}u(\kappa) - B_n - B_{n-1}\frac{\kappa^{(1)}}{\Gamma(2)} - B_{n-2}\frac{\kappa^{(2)}}{\Gamma(3)} - \cdots - B_1\frac{\kappa^{(n-1)}}{\Gamma(n)}$. *Then, we have*

$$G_a^n(\kappa) - G_a^n(a) = \frac{1}{(n-1)!}\sum_{s=a}^{\kappa-n}(\kappa - s - 1)^{(n-1)}u(s),\ \kappa \in J_1,\ \kappa - n - a \in \mathbb{N}(1).$$

Proof: *The proof is similar to Theorem* 3.2.7. *(Repalcing 0 by a in* (3.55)*)*

Corollary 3.2.12. $G_a^n(\kappa)$ *is the most general form of* $\Delta^{-n}u(\kappa)$ *with respect to* a, $\kappa - n - a \in \mathbb{N}(1)$ *and* $\kappa \in \mathbb{R} - \mathbb{Z}$. *If we denote* $F_a^n(\kappa) = G_a^n(\kappa) - G_a^n(a)$, *then we have*

$$F_a^n(\kappa) = \frac{1}{(n-1)!}\sum_{s=a}^{\kappa-n}(\kappa - s - 1)^{(n-1)}u(s),\ \kappa \in \mathbb{R},\ \kappa - a - n \in \mathbb{N}(1). \tag{3.59}$$

Example 3.2.13. *Taking* $u(\kappa) = 2^\kappa$, $\Delta^{-3}2^\kappa = 2^\kappa$, $2^0 \neq 0$ *in Example 3.2.6 and using the Theorem 3.2.7, we have*

$$G_0^3(\kappa) = \Delta^{-3}2^\kappa - A_3 - A_2\frac{\kappa^{(1)}}{1!} - A_1\frac{\kappa^{(2)}}{2!} \tag{3.60}$$

where $A_r = \Delta^{-r}2^\kappa\Big|_{\kappa=0}$, $r = 1, 2, 3$

$A_1 = \Delta^{-1}2^\kappa\Big|_{\kappa=0} = 2^\kappa\Big|_{\kappa=0} = 1$, $A_2 = \Delta^{-2}2^\kappa\Big|_{\kappa=0} = 2^\kappa\Big|_{\kappa=0} = 1$

and $A_3 = \Delta^{-3}2^\kappa\Big|_{\kappa=0} = 2^\kappa\Big|_{\kappa=0} = 1$

Substituting $A_r's$ *in equation (3.60), we get*

$$G_0^3(\kappa) = 2^\kappa - 1 - \frac{\kappa^{(1)}}{1!} - \frac{\kappa^{(2)}}{2!}.$$

As in the example 3.2.6, taking $\kappa = 5$, *we get*

$G_0^3(5) = 2^5 - 1 - \frac{5^{(1)}}{1!} - \frac{5^{(2)}}{2!} = 32 - 1 - 5 - \frac{5.4}{2} = 16.$

$G_0^3(0) = 2^0 - 1 - 0 = 0.$, *and*

$F_0^3(5) = 16 = \frac{1}{\Gamma(3)}\sum_{s=0}^{5}(5 - s - 1)^{(2)}2^s = G_0^{(3)}(5) - G_0^{(3)}(0).$

Hence (3.59) is verified for positive integer κ.

In the following theorem, we derive the result to a base'a'.

Theorem 3.2.14. *Let* $\kappa - n - a \in \mathbb{N}(0)$ *and* $f : J_1 \to R$ *be function and*
$F_a^n(\kappa) = \Delta^{-n}u(\kappa) - B_n - B_{n-1}\frac{(\kappa-a)^{(1)}}{1!} - B_{n-2}\frac{(\kappa-a)^{(2)}}{2!} \cdots -$
$B_1\frac{(\kappa-a)^{(n-1)}}{(n-1)!}, \kappa \in \mathbb{R}$, *where* $B_r = \Delta^{-r}u(\kappa)\Big|_{\kappa=a}$, . *Then, we have*

$$F_a^n(\kappa) = \frac{1}{(n-1)!}\sum_{s=a}^{\kappa-n}(\kappa - s - 1)^{(n-1)}u(s). \tag{3.61}$$

Proof: From the Equation (3.21), we have

$$\Delta^{-1}u(\kappa) - \Delta^{-1}u(\kappa-m) = u(\kappa-1) + u(\kappa-2) + u(\kappa-3) + \cdots + u(\kappa-m) \quad (3.62)$$

which can be expressed as

$\Delta^{-1}u(\kappa-1) - \Delta^{-1}u(\kappa-m) = u(\kappa-2) + u(\kappa-3) + \cdots + u(\kappa-m),$

$\Delta^{-1}u(\kappa-2) - \Delta^{-1}u(\kappa-m) = u(\kappa-3) + u(\kappa-4) + \cdots + u(\kappa-m),$

$\Delta^{-1}u(\kappa-3) - \Delta^{-1}u(\kappa-m) = u(\kappa-4) + u(\kappa-5) + \cdots + u(\kappa-m)$

and soon. Finally we get

$\Delta^{-1}u(\kappa-m+1) - \Delta^{-1}u(\kappa-m) = u(\kappa-m)$

$\Delta^{-1}u(\kappa-m) - \Delta^{-1}u(\kappa-m) = 0$

Adding above terms starting from $\Delta^{-1}u(\kappa-1)$, and replacing u by $\Delta^{-1}u$ in (3.62) and using it, we find that

$$\Delta^{-2}u(\kappa) - \Delta^{-2}u(\kappa-m) - m\Delta^{-1}u(\kappa-m)$$
$$= mu(\kappa-m) + (m-1)u(\kappa-m+1) + \cdots + 2u(\kappa-3) + 1u(\kappa-2). \quad (3.63)$$

By taking $\kappa - m = a$ and hence $\kappa - a = m$, (3.63) can be expressed as

$$\Delta^{-2}u(\kappa) - B_2 - \frac{(\kappa-a)^{(1)}}{1!}B_1 = \sum_{s=a}^{\kappa-2}(\kappa-s-1)^{(1)}u(s), \quad (3.64)$$

where $B_2 = \Delta^{-2}u(\kappa)\Big|_{\kappa=0}, B_1 = \Delta^{-2}u(\kappa)\Big|_{\kappa=a}$.

By induction and taking $\kappa \in \mathbb{R}, \kappa - n - a \in \mathbb{N}(0)$, we obtain

$$\Delta^{-n}u(\kappa) - B_n - B_{n-1}\frac{(\kappa-a)^{(1)}}{1!} - B_{n-2}\frac{(\kappa-a)^{(2)}}{2!} \cdots - B_1\frac{(\kappa-a)^{(n-1)}}{(n-1)!}$$

$$= \frac{1}{(n-1)!} \sum_{s=a}^{\kappa-n} (\kappa - s - 1)^{(n-1)} u(s), \tag{3.65}$$

where $B_r = \Delta^{-r} u(\kappa)\Big|_{\kappa=a}$, $r = 1, 2, \cdots n.$

If we define L.H.S of (3.65) *as* $F_a^n(\kappa)$,. *we find*

$$F_a^n(\kappa) = \frac{1}{(n-1)!} \sum_{s=a}^{\kappa-n} (\kappa - s - 1)^{(n-1)} u(s). \tag{3.66}$$

From (3.63), *we have*

$$\Delta^{-2}u(\kappa) - \Delta^{-2}u(\kappa-m) - m\Delta^{-1}u(\kappa-m) = (m-1)u(\kappa-m) + (m-1)u(\kappa-m+1)$$

$$+ (m-2)u(\kappa - m + 2) + \cdots + 2u(\kappa - 3) + 1u(\kappa - 2), \; m = \kappa - a$$

which can be expressed as

$$\Delta^{-2}u(\kappa) - \Delta^{-2}u(\kappa-m) - m\Delta^{-1}u(\kappa-m) = 1u(\kappa-2) + 2u(\kappa-3)$$

$$+ \cdots + (m-1)u(\kappa - m + 1) + (m-1)u(\kappa - m). \tag{3.67}$$

Replacing κ *by* $\kappa - 1$ *and* m *by* $m - 1$ *in* (3.67) *yields*

$$\Delta^{-2}u(\kappa-1) - \Delta^{-2}u(\kappa-m) - (m-1)\Delta^{-1}u(\kappa-m) = 1u(\kappa-3) + 2u(\kappa-4)$$

$$+ \cdots + (m-3)u(\kappa - m + 1) + (m-2)u(\kappa - m). \tag{3.68}$$

Similarly,

$$\Delta^{-2}u(t-2) - \Delta^{-2}u(t-m) - (m-2)\Delta^{-1}u(t-m) = 1u(\kappa-4) + 2u(\kappa-5)$$

$$+ \cdots + (m-4)u(\kappa - m + 1) + (m-3)u(\kappa - m),$$

$$\Delta^{-2}u(\kappa-3)-\Delta^{-2}u(\kappa-m)-(m-3)\Delta^{-1}u(t-m) = 1u(\kappa-5)+2u(\kappa-6)$$
$$+\cdots+(m-5)u(\kappa-m+1)+(m-4)u(\kappa-m),$$

and so on. Finally

$$\Delta^{-2}u(\kappa-m+1)-\Delta^{-2}u(\kappa-m)-1\Delta^{-1}u(\kappa-m) = 1u(\kappa-m).$$

$$\Delta^{-2}u(\kappa-m)-\Delta^{-2}u(\kappa-m) = 0.$$

Adding the above equations starting from (3.68), replacing u by $\Delta^{-2}u$ in (3.21) and using (3.12), we derive

$$\Delta^{-3}u(\kappa)-\Delta^{-3}u(\kappa-m)-\frac{m}{1!}\Delta^{-2}u(\kappa-m)-\frac{m^{(2)}}{2!}\Delta^{-1}u(\kappa-m) = \frac{2^{(2)}}{2!}u(\kappa-3)$$
$$+\frac{3^{(2)}}{2!}u(\kappa-4)+\frac{4^{(2)}}{2!}u(\kappa-5)+\cdots+\frac{(m-1)^{(2)}}{2!}u(\kappa-m),\ m=\kappa-a \qquad (3.69)$$

By induction hypothesis, we assume that (3.65) is true up-to $n-1$. That is,

$$\Delta^{-(n-1)}u(\kappa)-\Delta^{-(n-1)}u(\kappa-m)-\frac{m^{(1)}}{1!}\Delta^{-(n-2)}u(\kappa-m)-\frac{m^{(2)}}{2!}\Delta^{-(n-3)}u(\kappa-m)-$$
$$\cdots-\frac{m^{(n-2)}}{(n-2)!}\Delta^{-1}u(\kappa-m) = \frac{(n-2)^{(n-2)}}{(n-2)!}u(\kappa-n+1)+\frac{(n-1)^{(n-2)}}{(n-2)!}u(\kappa-n)$$
$$+\frac{(n)^{(n-2)}}{(n-2)!}u(\kappa-n-1)+\cdots+\frac{(m-1)^{(n-2)}}{(n-2)!}u(\kappa-m),\ m=\kappa-a,\ \kappa-a-n\in\mathbb{N}(0). \qquad (3.70)$$

Replacing κ by $\kappa-1$ and m by $m-1$ in (3.70) yields

$$\Delta^{-(n-1)}u(\kappa-1)-\Delta^{-(n-1)}u(\kappa-m)-\frac{(m-1)^{(1)}}{1!}\Delta^{-(n-2)}u(\kappa-m)-$$
$$\frac{(m-1)^{(2)}}{2!}\Delta^{-(n-3)}u(\kappa-m)-\cdots-\frac{(m-1)^{(n-2)}}{(n-2)!}\Delta^{-1}u(\kappa-m) = \frac{(n-2)^{(n-2)}}{(n-2)!}u(\kappa-n+1)$$

$$+\frac{(n-1)^{(n-2)}}{(n-2)!}u(\kappa-n)+\cdots+\frac{(m-2)^{(n-2)}}{(n-2)!}u(\kappa-m). \qquad (3.71)$$

Similarly,

$$\Delta^{-(n-1)}u(\kappa-2)-\Delta^{-(n-1)}u(\kappa-m)-\frac{(m-2)^{(1)}}{1!}\Delta^{-(n-2)}u(\kappa-m)-$$

$$\frac{(m-2)^{(2)}}{2!}\Delta^{-(n-3)}u(\kappa-m)-\cdots-\frac{(m-2)^{(n-2)}}{(n-2)!}\Delta^{-1}u(\kappa-m)=\frac{(n-2)^{(n-2)}}{(n-2)!}u(\kappa-n+1)$$

$$+\frac{(n-1)^{(n-2)}}{(n-2)!}u(\kappa-n)+\cdots+\frac{(m-2)^{(n-2)}}{(n-2)!}u(\kappa-m).$$

and so on. Finally, we arrive

$$\Delta^{-(n-1)}u(\kappa-m+1)-\Delta^{-(n-1)}u(\kappa-m)-\frac{1^{(1)}}{1!}\Delta^{-(n-2)}u(\kappa-m)-\frac{1^{(2)}}{2!}\Delta^{-(n-3)}u(\kappa-m)$$

$$-\cdots-\frac{1^{(n-2)}}{(n-2)!}\Delta^{-1}u(\kappa-m)=\frac{(n-2)^{(n-2)}}{(n-2)!}u(\kappa-m).$$

$$\Delta^{-(n-1)}u(\kappa-m)-\Delta^{-(n-1)}u(\kappa-m)=0.$$

Adding all the above relations starting from (3.71), replacing u by $\Delta^{-(n-1)}u$ in (3.21) and using (3.12), we arrive our required result as below:

$$\Delta^{(n-1)}u(\kappa-1)+\Delta^{(n-1)}u(\kappa-2)+\cdots+\Delta^{(n-1)}u(\kappa-m)-m\Delta^{-(n-1)}u(\kappa-m)$$
$$-\left\{\frac{(m-1)^{(1)}}{1!}+\frac{(m-2)^{(1)}}{1!}+\cdots+\frac{1^{(1)}}{1!}\right\}\Delta^{-(n-2)}u(\kappa-m)$$
$$-\left\{\frac{(m-1)^{(2)}}{2!}+\frac{(m-2)^{(2)}}{2!}+\cdots+\frac{1^{(2)}}{2!}\right\}\Delta^{-(n-3)}u(\kappa-m)-\cdots$$
$$-\left\{\frac{(m-1)^{(n-2)}}{(n-2)!}+\frac{(m-2)^{(n-2)}}{(n-2)!}+\cdots+\frac{1^{(n-2)}}{(n-2)!}\right\}\Delta^{-1}u(\kappa-m)$$
$$=\{\frac{(m-2)^{(n-2)}}{(n-2)!}+\frac{(m-3)^{(n-2)}}{(n-2)!}+\cdots+\frac{(n-2)^{(n-2)}}{(n-2)!}\}u(\kappa-m)+$$
$$\{\frac{(m-3)^{(n-2)}}{(n-2)!}+\frac{(m-4)^{(n-2)}}{(n-2)!}+\cdots+\frac{(n-2)^{(n-2)}}{(n-2)!}\}\Delta^{-1}u(\kappa-(m-1))+$$
$$\{\frac{(m-4)^{(n-2)}}{(n-2)!}+\frac{(m-5)^{(n-2)}}{(n-2)!}+\cdots+\frac{(n-2)^{(n-2)}}{(n-2)!}\}\Delta^{-1}u(\kappa-(m-2))+$$

$$\cdots + \frac{(n-2)^{(n-2)}}{(n-2)!}u(\kappa-n-1),$$

From (3.12) *and* $(n-2)^{(n-1)} = 0$ *and* (3.21) *the above expression becomes*

$$\Delta^{-n}u(\kappa) - \Delta^{-n}u(\kappa-m) - \frac{m^{(1)}}{1!}\Delta^{-(n-1)}u(\kappa-m) - \frac{m^{(2)}}{2!}\Delta^{-(n-2)}u(\kappa-m)$$
$$-\frac{m^{(3)}}{3!}\Delta^{-(n-3)}u(\kappa-m) - \cdots - \frac{m^{(n-1)}}{(n-1)!}\Delta^{-1}u(\kappa-m) = \frac{(m-1)^{(n-1)}}{(n-1)!}u(\kappa-m)$$
$$+\frac{(m-2)^{(n-1)}}{(n-1)!}u(\kappa-m+1)+\frac{(m-3)^{(n-1)}}{(n-1)!}u(\kappa-m+2)+\cdots+\frac{(n-1)^{(n-1)}}{(n-1)!}u(\kappa-n-1). \tag{3.72}$$

By taking $\kappa - m = a,\ \kappa - a - n \in \mathbb{N}(0)$, (3.72) *can be expressed as*

$$\Delta^{-1}u(\kappa) - B_n - \frac{(\kappa-a)^{(1)}}{1!}B_{n-1} - \frac{(\kappa-a)^{(2)}}{2!}B_{n-2} - \cdots - \frac{(\kappa-a)^{(n-1)}}{(n-1)!}B_1$$
$$= \frac{(\kappa-a-1)^{(n-1)}}{(n-1)!}u(a) + \frac{(\kappa-a-2)^{(n-1)}}{(n-1)!}u(a+1) + \cdots + \frac{(n-1)^{(n-1)}}{(n-1)!}u(\kappa-n-1), \tag{3.73}$$

where $B_r = \Delta^{-r}u(\kappa-m),\ r = 1,2,3,\cdots,n$ *and which completes the proof.*

In the following theorem, we discuss higher order delta invere of constant function $u(\kappa) = 1$.

Example 3.2.15. *Consider the function* $u(\kappa) = 4^\kappa$. *Then* $\Delta 4^\kappa = 3.4^\kappa$ *yields* $\Delta^{-r}u(\kappa) = 3^{-r}4^\kappa,\ r = 1,2,\cdots$.

Taking $\kappa = 4.5,\ a = 1.5,\ n = 3$, *we get*

$$B_1 = \Delta^{-1}4^\kappa\Big|_{\kappa=1.5} = \left(\frac{1}{3}\right)4^{(1.5)} = \frac{8}{3}$$
$$B_2 = \Delta^{-2}4^\kappa\Big|_{\kappa=1.5} = \left(\frac{1}{9}\right)4^{(1.5)} = \frac{8}{9}$$
$$B_3 = \Delta^{-3}4^\kappa\Big|_{\kappa=1.5} = \left(\frac{1}{27}\right)4^{(1.5)} = \frac{8}{27}$$

$$F^{(3)}_{1.5}(\kappa) = \Delta^{-3}4^{\kappa} - B_3 - B_2\frac{(\kappa-a)^{(1)}}{1!} - B_1\frac{(\kappa-a)^{(2)}}{2!}$$
$$F^{(3)}_{1.5}(4.5) = \frac{4^{4.5}}{27} - \frac{8}{27} - \frac{8}{9}.\frac{3^{(1)}}{1!} - \frac{8}{3}.\frac{3^{(2)}}{2!} = \frac{4^4}{27}.2 - \frac{8}{27} - \frac{8}{9}.3 - \frac{8}{3}\frac{3.2}{2!}$$
$$= \frac{8}{27}[4^3-1-9] - 8 = 8\times 2 - 8 = 8 = \frac{1}{2!}\sum_{s=1.5}^{4.5}(4.5-s-1)^{(2)}4^s.$$

Hence, (3.67) is verified.

Example 3.2.16. *Consider the constant function $u(\kappa)=1,\ \kappa\in\mathbb{R}$.*

Since $\kappa^{(0)}=1$, we have $u(\kappa)=\kappa^{(0)},\ u(0)=1,\ \kappa^{(0)}=1,\ a=0$.

$$\Delta^{-1}(1) = \Delta^{-1}\kappa^{(0)} = \frac{\kappa^{(1)}}{1!} \Rightarrow A_1 = 0.$$
$$\Delta^{-2}(1) = \Delta^{-2}\kappa^{(0)} = \frac{\kappa^{(2)}}{2!} \Rightarrow A_2 = 0.$$
$$\Delta^{-3}(1) = \Delta^{-3}\kappa^{(0)} = \frac{\kappa^{(3)}}{3!} \Rightarrow A_3 = 0.$$
$$G^3_0(\kappa) = \frac{\kappa^{(3)}}{3!} - \frac{\kappa^{(2)}}{2!},\ G^3_0(0)=0,\ \kappa=10.$$
$$G^3_0(10) = \frac{10^{(3)}}{3!} - \frac{10^{(2)}}{2!} = \frac{10.9.8}{3.2} - \frac{10.9}{2} = 120.$$
$$F^3_0(10) = \frac{1}{2}\sum_{s=0}^{7}\frac{\Gamma(10-s)}{\Gamma(10-s-2)}u(s) = \frac{1}{2}\sum_{s=0}^{7}\frac{(9-s)!}{(7-s)!}.1$$
$$= \frac{1}{2}\sum_{s=0}^{7}(9-s)(8-s)$$
$$= 9.4+4.7+7.3+3.5+5.2+2.3+3.1+1$$
$$= 36+28+21+15+10+6+3+1 = 120.$$

In the following example we assume the origin ‘a’=0.

Example 3.2.17. *Let $u(\kappa)=1$, for all $\kappa\in\mathbb{R}$, $a=1.5$ and using (3.4), we have*

$$\Delta^{-1}(1) = \Delta^{-1}(\kappa-a)^{(0)} = \frac{(\kappa-a)^{(1)}}{1!},\ B_1 = \frac{(\kappa-a)^{(1)}}{1!}\Big|_{\kappa=a} = 0$$
$$\Delta^{-2}(1) = \Delta^{-2}(\kappa-a)^{(0)} = \frac{(\kappa-a)^{(2)}}{2!},\ B_2 = \frac{(\kappa-a)^{(2)}}{2!}\Big|_{\kappa=a} = 0$$

$$\Delta^{-3}(1) = \Delta^{-3}(\kappa - a)^{(0)} = \frac{(\kappa - a)^{(3)}}{3!}, \; B_3 = \frac{(\kappa - a)^{(3)}}{3!}\Big|_{\kappa=a} = 0$$

By (3.65), we have

$$\Delta^{-3}(1) - B_3 - B_2\frac{(\kappa - a)^{(1)}}{1!} - B_1\frac{(\kappa - a)^{(1)}}{1!} = \frac{1}{2}\sum_{s=a}^{\kappa-3}(\kappa - s - 2)^{(0)}(1).$$

$$\frac{(\kappa - a)^{(3)}}{3!} = \frac{1}{2}\sum_{s=a}^{\kappa-3}(\kappa - s - 1)(\kappa - s - 2), \; t - a \in \mathbb{N}(0). \tag{3.74}$$

If we take $\kappa = 8.5$, $a = 1.5$, (3.74) *becomes* $\frac{(8.5 - 1.5)^{(3)}}{3!} = \frac{1}{2}\sum_{s=1.5}^{5.5}(7.5 - s)(6.5 - s)$.
$\frac{7^{(3)}}{3!} = \frac{1}{2}[6(5) + 5(4) + 4(3) + 3(2) + 2(1)] = 3(5) + 5(2) + 2(3) + 3 + 1 = 35.$

Remark 3.2.18. *From the property of Gamma function, we have*

$\Gamma(\kappa + 1) = \kappa(\kappa - 1)(\kappa - 2)\cdots(\kappa - r + 1)\Gamma(\kappa - r + 1)$,*which yields*
$\kappa^{(r)} = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - r + 1)}$ *if* $r \in \mathbb{N}(1)$.

Definition 3.2.19. *The falling factorial for real index* ν *is defined by*

$$\kappa^{(\nu)} = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa + 1 - \nu)}, \; \kappa + 1 - \nu \notin \{0, -1, -2, \cdots\}.$$

Theorem 3.2.14 motivates us to find the exact form of LHS of (3.1).

Let u be a real valued function, κ and a belongs to domain of u such that $\kappa - a \in \mathbb{N}(0)$.

If there exists a real valued function F_a^ν, depending on a and $\nu > 0$ such that

$$F_a^\nu(\kappa) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{\kappa-\nu}\frac{\Gamma(\kappa - s)}{\Gamma(\kappa - s - (\nu - 1))}u(s), \tag{3.75}$$

then $F_a^\nu(\kappa)$ is called as exact form of ν^{th} fractional sum of $u(\kappa)$ based at a.

(i) If $u(\kappa) = 1, \kappa \in \mathbb{R}$ and $\nu = n \in \mathbb{N}(1)$ then Example 3.2.17 shows that

$$F_a^n(\kappa) = \frac{(\kappa - a)^{(n)}}{\Gamma(n+1)} = \frac{1}{\Gamma(n)} \sum_{s=a}^{\kappa-n} \frac{\Gamma(\kappa - s)}{\Gamma(\kappa - s - (n-1))} u(\kappa).$$

(ii) If $u(\kappa) = \kappa^{(\nu)}, \kappa \in \mathbb{R}$ and $\nu = n \in \mathbb{N}(1)$ then

$$F_a^n(\kappa) = \frac{(\kappa - a)^{(\nu+n)}}{(\nu+n)^{(n)}} = \frac{1}{\Gamma(n)} \sum_{s=a}^{\kappa-n} \frac{\Gamma(\kappa - s)}{\Gamma(\kappa - s - (n-1))} u(s).$$

(iii) If $u(\kappa) = (\kappa - a)^{(\nu)}, \kappa \in \mathbb{R}$, $\kappa - a \in \mathbb{N}(1)$ and $\nu = n \in \mathbb{N}(1)$ then

$$F_a^n(\kappa) = \frac{(\kappa - a)^{(\nu+n)}}{(\nu+n)^{(n)}} = \frac{1}{\Gamma(n)} \sum_{s=a}^{\kappa-n} \frac{\Gamma(\kappa - s)}{\Gamma(\kappa - s - (n-1))} (s-a)^{(\nu)}.$$

3.3 Higher Order Alpha-Delta Operator

The forward difference or operators is applicable in solving the problems in mathematical sciences, physical sciences, life sciences, scientific engineering. The numerical solution of m-th order difference equation is $\Delta_\ell^m v(\kappa) = u(\kappa)$, when $v(0) = 0$ is obtained by

$$\Delta_\ell^{-m} u(\kappa)\Big|_0^t = \sum_{r=0}^{s-m} \frac{\Gamma(m+r)}{\Gamma(r+1)\Gamma(m)} u(\kappa - (m+r)\ell), \tag{3.76}$$

where $\Delta_\ell u(\kappa) = u(\kappa + \ell) - u(\kappa)$, and Γ is a Gamma function.

It is also possible to develop fractional order anti-difference corresponding to equation (3.76) by replacing the integer m into real number $\nu > 0$. The corresponding numerical solution for ν-th order alpha-difference equation $\Delta_{\alpha,\ell} v(\kappa) =$

$u(\kappa)$ has been developed by many authors. When $\alpha = 1$, $\ell = 1$, the alpha delta operator becomes the usual forward difference operator Δ. For more details on alpha difference operator and its inverse one can refer [1, 6, 8, 34].

3.4 Finite Fractional Order Difference

In this section, first we present anti-difference Δ, Δ_ℓ and $\Delta_{\alpha(\ell)}$ for arriving at the general formula for numerical solution of fractional difference equation $\Delta^\nu v(\kappa) = u(\kappa)$, when $v(0) = \overset{\nu}{\Delta} u(t)\big|_{t=0} = 0$.

Lemma 3.4.1. *For any positive integer n, we have*

$$1^{(n)} + 2^{(n)} + 3^{(n)} + \cdots + \kappa^{(n)} = \frac{(\kappa+1)^{(n+1)}}{n+1}, \tag{3.77}$$

where $\kappa^{(n)} = \prod_{r=0}^{n-1} (\kappa - r)$.

Proof. The proof follows by induction method. □

Theorem 3.4.2. *Let $\kappa = s\ell$, $0 < \ell < \infty$, $m < s$, $u(0) = 0$ and $m \in \mathbb{N}(1)$, Then*

$$\Delta_\ell^{-m} u(\kappa) = \sum_{r=0}^{s-m} \frac{\Gamma(m+r)}{\Gamma(r+1)\Gamma(m)} u(\kappa - (m+r)\ell). \tag{3.78}$$

Proof. Let ℓ be any real in $(0, \infty)$. Then, $\Delta_\ell u(\kappa) = u(\kappa + \ell) - u(\kappa)$

Since $\kappa = s\ell$, where $s \in \mathbb{N}(0)$, we take

$$v(\kappa) = u(\kappa - \ell) + u(\kappa - 2\ell) + u(\kappa - 3\ell) + \cdots + u(0) \tag{3.79}$$

$$v(\kappa+\ell) = u(\kappa) + u(\kappa-\ell) + u(\kappa-2\ell) + \cdots + u(0) \tag{3.80}$$

(3.79) − (3.80) $\Rightarrow \Delta_\ell v(\kappa) = u(\kappa)$, which gives $\Delta_\ell^{-1}u(\kappa) = v(\kappa)$,

where $v(\kappa)$ is given in (3.79).

$$\textit{Hence } \Delta_\ell^{-1}u(\kappa) = u(\kappa-\ell) + u(\kappa-2\ell) + u(\kappa-3\ell) + \cdots + u(0). \tag{3.81}$$

Taking Δ_ℓ^{-1} on both sides,

$$\Delta_\ell^{-2}u(\kappa) = \Delta_\ell^{-1}u(\kappa-\ell) + \Delta_\ell^{-1}u(\kappa-2\ell) + \Delta_\ell^{-1}u(\kappa-3\ell) + \cdots + \Delta_\ell^{-1}u(0)$$

By applying (3.81) for $\kappa-\ell$, we get

$$\begin{aligned}\Delta_\ell^{-2}u(\kappa) = {} & u(\kappa-2\ell) + u(\kappa-3\ell) + u(\kappa-4\ell) + \cdots + u(0) + \\ & + u(\kappa-3\ell) + u(\kappa-4\ell) + u(\kappa-5\ell) + \cdots + u(0) + \\ & + u(\kappa-4\ell) + u(\kappa-5\ell) + u(\kappa-6\ell) + \cdots + u(0) + \\ & \vdots \\ & + u(3\ell) + u(2\ell) + u(\ell) + u(0) + \\ & + u(2\ell) + u(\ell) + u(0) + u(\ell) + u(0) + u(0)\end{aligned}$$

Grouping the terms, we find that

$$\Delta_\ell^{-2}u(\kappa) = \frac{1^{(1)}}{1!}u(\kappa-2\ell) + \frac{2^{(1)}}{1!}u(\kappa-3\ell) + \cdots + \frac{(s-1)^{(2)}}{1!}u(0), \text{ where } \kappa - s\ell = 0$$

Again taking Δ_ℓ^{-1} on both sides and by using (3.77), we get

$$\Delta_\ell^{-3}u(\kappa) = \frac{2^{(2)}}{1!}u(\kappa-3\ell) + \frac{3^{(2)}}{1!}u(\kappa-4\ell) + \cdots + \frac{(s-1)^{(2)}}{1!}u(0), \text{ where } \kappa - s\ell = 0$$

Proceeding like this, we arrive

$$\begin{aligned}\Delta_\ell^{-m}u(\kappa) = {} & \frac{(m-1)^{(m-1)}}{(m-1)!}u(\kappa-m\ell) + \frac{m^{(m-1)}}{(m-1)!}u(\kappa-(m+1)\ell) \\ & + \cdots + \frac{(s-1)^{(m-1)}}{(m-1)!}u(0), \text{ where } \kappa - s\ell = 0\end{aligned}$$

$$= \frac{\Gamma(m)}{\Gamma(m-(m-1))\Gamma(m)} u(\kappa - m\ell) + \frac{\Gamma(m+1)}{\Gamma(m+1-(m-1))\Gamma(m)}$$
$$\times\, u(\kappa-(m+1)\ell) + \frac{\Gamma(m+(s-m))}{\Gamma(m+(s-m)-(m-1))\Gamma(m)} u(0),$$

which gives (3.78). □

Corollary 3.4.3. *Assume that* $\Delta v(\kappa) = u(\kappa)$ *and v(0)=0. The m-th order delta inverse of* $u(\kappa)$ *is defined as*

$$\Delta^{-m} u(\kappa) = \sum_{r=0}^{\kappa-m} \frac{(m-1+r)^{(m-1)}}{(m-1)!} u(\kappa-(m+r)), n \in \mathbb{N}(m). \tag{3.82}$$

Proof. The proof follows by taking $\ell = 1$ in Theorem 4.2.2. □

Corollary 3.4.4. *Let* $\kappa > m\ell$, $0 < \ell < \infty$, $u(\kappa - m\ell) \in \mathbb{N}$, *Then*

$$\Delta_\ell^{-\nu} u(\kappa)\Big|_{\kappa-m\ell}^{\kappa} = \sum_{r=0}^{m} \frac{(\nu+r)^{(\nu)}}{\ell^\nu} u(\kappa-\ell-r\ell), \tag{3.83}$$

where $n^{(\nu)} = \dfrac{\Gamma(n+1)}{\Gamma(n+1-\nu)}$.

3.5 Higher order ℓ- Delta operator

In this section, we arrive at the general form of fractional order delta operator on trigonometric function.

Theorem 3.5.1. *For positive real m and* $\ell \neq 0$, *we have*

$$\Delta_\ell^m \sin\kappa = 2^m \sin^m(\frac{\ell}{2}) \sin(\frac{m\pi}{2} + \frac{m\ell}{2} + \kappa) \tag{3.84}$$

Proof. By the linear operator Δ_ℓ on $\sin\kappa$, we have $\Delta_\ell \sin\kappa = \sin(\kappa+\ell) - \sin\kappa$ which is similar to $\Delta_\ell \sin\kappa = 2\cos(\frac{\kappa+\ell+\kappa}{2})\sin(\frac{\kappa+\ell-\kappa}{2}) = 2\cos(\kappa+\frac{\ell}{2})\sin(\frac{\ell}{2})$. Thus,

$$\Delta_\ell \sin\kappa = 2\sin(\frac{\ell}{2})\sin(\frac{\pi}{2}+\frac{\ell}{2}+\kappa). \tag{3.85}$$

Taking Δ_ℓ again on both sides of (3.85), we get

$$\Delta_\ell^2 \sin\kappa = 2\sin(\frac{\ell}{2})\Delta_\ell \sin(\frac{\pi}{2}+\frac{\ell}{2}+\kappa).$$

While solving the above relation, we obtain

$$\Delta_\ell^2 \sin\kappa = 2^2\sin^2(\frac{\ell}{2})\sin(2\frac{\pi}{2}+2\frac{\ell}{2}+\kappa). \tag{3.86}$$

Again taking Δ_ℓ again on both sides of (3.86), we get

$$\Delta_\ell^3 \sin\kappa = 2^3\sin^3(\frac{\ell}{2})\sin(\frac{3\pi}{2}+\frac{3\ell}{2}+\kappa). \tag{3.87}$$

Proceeding the steps up-to m times, we find

$$\Delta_\ell^m \sin\kappa = 2^m\sin^m(\frac{\ell}{2})\sin(\frac{m\pi}{2}+\frac{m\ell}{2}+\kappa). \tag{3.88}$$

□

Theorem 3.5.2. *For positive integer m and $\ell \neq 0$, we have*

$$\Delta_\ell^m \cos\kappa = 2^m\sin^m(\frac{\ell}{2})\cos(\frac{m\pi}{2}+\frac{m\ell}{2}+\kappa). \tag{3.89}$$

Proof. The proof is similar to Theorem 3.5.1. □

Example 3.5.3. *Taking* $\nu = 20$ *in* (3.84) *and* (3.89), *we get*

$\Delta_\ell^{20}(\sin\kappa) = 2^{20}\sin^{20}(\frac{\ell}{2})\sin(\frac{20\pi}{2} + \frac{20\ell}{2} + \kappa)$

$\Delta_\ell^{20}(\cos\kappa) = 2^{20}\sin^{20}(\frac{\ell}{2})\cos(\frac{20\pi}{2} + \frac{20\ell}{2} + \kappa)$

Remark: We can extend the relations (3.84) and (3.89) to negative integers also.

3.6 Inverse of Higher Order Alpha Delta Operator

In this section, we develop the theory of higher order of alpha delta operator and its sums on trigonometric functions.

Definition 3.6.1. *Let* $\ell > 0$ *and* u, v *be two functionsand* $\alpha \neq 0$. *Then the alpha delta operator* $\Delta_{\alpha,\ell}$ *on* $u(\kappa)$ *is defined by*

$$\Delta_{\alpha,\pm\ell}u(\kappa) = u(\kappa \pm \ell) - \alpha u(\kappa). \tag{3.90}$$

If $\Delta_{\alpha,\pm\ell}v(\kappa) = u(\kappa)$, *then the inverse is* $\Delta^{-1}_{\alpha,\pm\ell}u(\kappa) = v(\kappa) + c$, c *is constant and*

$$\Delta^{-1}_{\alpha,\pm\ell}u(\kappa)\big|_a^d = v(d) - \alpha v(a). \tag{3.91}$$

Theorem 3.6.2. *The higher order of alpha delta operator on* $\sin\kappa$ *is obtained as*

$$\Delta^m_{\alpha,\ell}\sin\kappa = \sin^m\ell\sin\left(\frac{m\pi}{2} + \kappa\right), \alpha = \cos\ell. \tag{3.92}$$

Proof. From the definition of $\Delta_{\alpha,\ell}$, we have

$\Delta_{\alpha,\ell} \sin\kappa = \sin(\kappa+\ell) - \alpha \sin\kappa$ and taking $\alpha = \cos\ell$, we find

$$\Delta_{\alpha,\ell} \sin\kappa = \sin\kappa\cos\ell + \cos\kappa\sin\ell - \cos\ell\sin\kappa = \cos\kappa\sin\ell$$

$$= \sin\ell\sin\left(\frac{\pi}{2}+\kappa\right).$$

Similarly, $\Delta^2_{\alpha,\ell} \sin\kappa = \sin^2_\ell \sin(\frac{2\pi}{2}+n)$.

In general, $\Delta^m_{\alpha,\ell} \sin\kappa = \sin^m\ell\sin\left(\frac{m\pi}{2}+\kappa\right)$, which is (3.92). □

Theorem 3.6.3. *The higher order alpha delta operator on* $\cos\kappa$ *is obtained as*

$$\Delta^m_{\alpha,\ell} \cos\kappa = \sin^m\ell\cos\left(\frac{m\pi}{2}+\kappa\right), \alpha = \cos\ell. \tag{3.93}$$

Proof. Since $\Delta_{\alpha,\ell}\cos\kappa = \cos(\kappa+\ell) - \alpha\cos\kappa$ and taking $\alpha = \cos\ell$,

$$\Delta_{\alpha,\ell}\cos\kappa = \cos\kappa\cos\ell - \sin\kappa\sin\ell - \cos\ell\cos\kappa = -\sin\kappa\sin\ell$$

$$= \sin\ell\cos\left(\frac{\pi}{2}+\kappa\right).$$

$$\Delta^2_{\alpha,\ell}\cos\kappa = \sin^2\ell\cos\left(\frac{2\pi}{2}+\kappa\right).$$

In general, $\Delta^m_{\alpha,\ell}\cos\kappa = \sin^m\ell\cos\left(\frac{m\pi}{2}+\kappa\right)$, which is (3.93). □

Theorem 3.6.4. *The higher order alpha delta operator on* $\sin a\kappa$ *is*

$$\Delta^m_{\alpha,\ell}\sin a\kappa = \sin^m a\ell\sin a\Big(\frac{m\pi}{2}+\kappa\Big), \alpha = \cos\ell, m \in \mathbb{N}(1). \tag{3.94}$$

Proof. Since $\Delta_{\alpha,\ell}\sin a\kappa = \sin a(\kappa+\ell) - \alpha\sin a\kappa$, and taking $\alpha = \cos a\ell$

$$\Delta_{\alpha,\ell}\sin a\kappa = \sin a\kappa\cos a\ell + \cos a\kappa\sin a\ell - \alpha\sin a\kappa = \cos a\kappa\sin a\ell$$

$$= \sin a\ell\sin a\Big(\frac{\pi}{2}+\kappa\Big).$$

$$\Delta^2_{\alpha,\ell} \sin a\kappa = \sin^2 a\ell \sin a(\frac{2\pi}{2} + n).$$

In general, $\Delta^m_{\alpha,\ell} \sin a\kappa = \sin^m a\ell \sin a\Big(\frac{m\pi}{2} + \kappa\Big)$, which is (3.94). □

Theorem 3.6.5. *The higher order alpha deltaa operator on* $\cos a\kappa$ *is*

$$\Delta^m_{\alpha,\ell} \cos a\kappa = \sin^m a\ell \cos a\Big(\frac{m\pi}{2} + \kappa\Big), \alpha = \cos\ell. \tag{3.95}$$

Proof. From $\Delta_{\alpha,\ell} \cos a\kappa = \cos a(\kappa + \ell) - \alpha \cos a\kappa$ and $\alpha = \cos a\ell$, we have

$$\Delta_{\alpha,\ell} \cos a\kappa = \cos a\kappa \cos a\ell - \sin a\kappa \sin a\ell - \cos a\ell \cos a\kappa = -\sin a\kappa \sin a\ell$$

$$= \sin a\ell \cos a\Big(\frac{\pi}{2} + \kappa\Big).$$

$$\Delta^2_{\alpha,\ell} \cos \kappa = \sin^2 a\ell \cos a\Big(\frac{2\pi}{2} + \kappa\Big).$$

In general, $\Delta^m_{\alpha,\ell} \cos a\kappa = \sin^m a\ell \cos a\Big(\frac{m\pi}{2} + \kappa\Big)$, which is (3.95). □

Theorem 3.6.6. *The higher order inverse of alpha-delta operator on the sine function is*

$$\Delta^{-m}_{\alpha,\ell} \sin \kappa = \sin^{-m} \ell \sin\left(\kappa - \frac{m\pi}{2}\right), \alpha = \cos\ell. \tag{3.96}$$

Proof. From $\Delta^m_{\alpha,\ell} \sin \kappa = \sin^m \ell \sin(\frac{m\pi}{2} + \kappa)$ and $m = -m$,

$$\Delta^{-m}_{\alpha,\ell} \sin \kappa = \frac{\sin\left(\kappa - \frac{m\pi}{2}\right)}{\sin^m \ell}$$

$$\Delta^m_{\alpha,\ell} \sin\left(\kappa - \frac{m\pi}{2}\right) = \sin\kappa \sin^m \ell.$$

Similarly $\Delta^{-m}_{\alpha,\ell} \sin \kappa = \sin^{-m} \ell \sin\left(\kappa - \frac{m\pi}{2}\right)$, which is (3.96). □

Theorem 3.6.7. *For any real function* $u(\kappa)$ *defined on* R *and* $m \in \mathbb{N}(1)$

$$\Delta^{-1}_{\alpha,\ell} u(\kappa) - \alpha^m \Delta^{-1}_{\alpha,\ell} u(\kappa - m\ell) = \sum_{r=1}^{m} \alpha^{r-1} u(\kappa - r\ell). \tag{3.97}$$

Proof. From the definition of inverse of alpha delta operator we have,

$$\Delta_{\alpha,\ell}^{-1}u(\kappa) = v(\kappa), \quad v(\kappa+\ell) = u(\kappa) + \alpha v(\kappa). \tag{3.98}$$

Replacing κ by $\kappa-\ell$, $\kappa-2\ell$, $\kappa-3\ell, \cdots$, $\kappa-m\ell$ in (3.98),

$$v(\kappa) = u(\kappa-\ell) + \alpha u(\kappa-2\ell) + \alpha^2 u(\kappa-3\ell) + \alpha^3 u(\kappa-4\ell) + \cdots$$
$$+ \alpha^{m-1}u(\kappa-m\ell) + \alpha^m v(\kappa-m\ell)$$

$$v(\kappa) - \alpha^m v(\kappa-m\ell) = u(\kappa-\ell) + \alpha u(\kappa-2\ell) + \alpha^2 u(\kappa-3\ell) + \alpha^3 u(\kappa-4\ell)$$
$$+ \cdots + \alpha^{m-1}u(\kappa-m\ell)$$

$$\Delta_{\alpha,\ell}^{-1}u(\kappa) - \alpha^m\Delta_{\alpha,\ell}^{-1}u(\kappa-m\ell) = u(\kappa-\ell) + \alpha u(\kappa-2\ell) + \alpha^2 u(\kappa-3\ell)$$
$$+ \alpha^3 u(\kappa-4\ell) + \cdots + \alpha^{m-1}u(\kappa-m\ell)$$

which gives (3.97). □

Example 3.6.8. *If $u(\kappa) = \sin\kappa$, then the equation (3.97) becomes*

$$\Delta_{\alpha,\ell}^{-1}\sin\kappa - \alpha^m\Delta_{\alpha,\ell}^{-1}\sin(\kappa-m\ell) = \textstyle\sum_{r=1}^{m}\alpha^{r-1}\sin(\kappa-r\ell)$$

Taking $\kappa = 5, \alpha = \cos\ell, \ell = 1, m = 2$, we have

$$\Delta_{\alpha,\ell}^{-1}\sin\kappa - (\cos\ell)^m\Delta_{\alpha,\ell}^{-1}\sin(\kappa-m\ell)$$
$$= \sin^{-1}(1)\sin(5-\tfrac{\pi}{2}) - \sin^{-1}(1)(\cos(1))^2\sin(3-\tfrac{\pi}{2})$$
$$= \sin^{-1}(1)\Big[\sin(5-\tfrac{\pi}{2}) - (\cos(1))^2\sin(3-\tfrac{\pi}{2})\Big]$$

$$\sin(4) + \cos(1)\sin(3) = \frac{\sin(-85) - \cos^2(1)\sin(-87)}{\sin(1)} \Rightarrow 0.1221 = 0.1221.$$

Thus, we extend the theory developed in chapter 3 to the ℓ- alpha delta operator and arriv at the higher order alpha delta operator and summation formula on trigonometric functions.

Chapter 4

Properties of Extorial Function

In this chapter, we introduce a new function called extorial function. Also we arrive as certain results involving the above function and its properties.

4.1 The ℓ - Extorial function

The newly defined ℓ-Extorial function is arrived by replacing the polynomial κ^n by polynomial factorial function $\kappa_\ell^{(n)}$ in the exponential function e^κ. The formal definition of extorial function is given below.

Definition 4.1.1. *The ℓ-extorial function denoted as $e(\kappa_\ell^{(n)})$ is defined as*

$$e(\kappa_\ell^{(n)}) = 1 + \frac{\kappa_\ell^{(n)}}{1!} + \frac{\kappa_\ell^{(2n)}}{2!} + \frac{\kappa_\ell^{(3n)}}{3!} + \cdots + \infty, \tag{4.1}$$

where $|\ell| \leq 1$ and $n, \kappa \in \mathbb{R}$.

Lemma 4.1.2. *[15] Let $|\ell| \leq 1$ and κ a real variable. Then the following holds.*

(i) $e(\kappa_0^{(1)}){=}e^{\kappa}$, *(ii)* $e((-\kappa)_1^{(1)}) = -\infty$, *(iii)* $e(\kappa_{-1^{(1)}}) = \infty$,

(iv) $e((-\kappa)_\ell^{(1)}) = 1 - \frac{\kappa_{-\ell}^{(1)}}{1!} + \frac{\kappa_{-\ell}^{(2)}}{2!} - \frac{\kappa_{-\ell}^{(3)}}{3!} + \cdots + \infty$,

(v) $e((-\kappa)_{-\ell}^{(1)}) = 1 - \frac{\kappa_\ell^{(1)}}{1!} + \frac{\kappa_\ell^{(2)}}{2!} - \frac{\kappa_\ell^{(3)}}{3!} + \cdots + \infty$,

(vi) $\Delta_\ell e(\kappa_\ell^{(1)}) = \ell e(\kappa_\ell^{(1)})$, *(vii)* $\Delta_\ell^n e(\kappa_\ell^{(n)}) = \ell^n e(\kappa_\ell^{(1)})$.

Lemma 4.1.3. *[15] Let κ be the multiple of ℓ. Then $e(\kappa_\ell^{(1)})$ can be expressed as finite series such that (i)* $e(\kappa_\ell^{(1)}) = \sum_{r=0}^{a} \frac{\kappa_\ell^{(r)}}{r!}$.

(ii) For any $\ell \in \mathbb{N}$, $e(-\ell)_{(-\ell)}^{(1)} = 1 - \ell$ and (iii) For $\kappa_1, \kappa_2 \in \mathbb{R}$ and $\ell \in (0, 1)$,

$$e(\kappa_1 + \kappa_2)_\ell^{(1)} = e(\kappa_1)_\ell^{(1)} e(\kappa_2)_\ell^{(1)}. \tag{4.2}$$

By expanding the terms and making simplification, we get the proof.

4.2 The ℓ- Extorial for Negative Index

In this section, we define extorial function for negative index and find relation among the delta operators.

Definition 4.2.1. *If $\kappa_\ell^{(rn)} \neq 0$ for $n > 0$ and $r \in \mathbb{N}$, then the negative index extorial function is defind as*

$$e(\kappa_\ell^{(-n)}) = 1 + \frac{1}{1!}\frac{1}{\kappa_\ell^{(n)}} + \frac{1}{2!}\frac{1}{\kappa_\ell^{(2n)}} + \frac{1}{3!}\frac{1}{\kappa_\ell^{(3n)}} + \cdots \infty \tag{4.3}$$

Remark 4.2.2. *(i)*$e(1_{-1}^{(-1)}) = \sum_{r=0}^{\infty} \frac{1}{(r!)^2}$, *(ii)*$e(-1_1^{(-1)}) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{(r!)^2}$, *(iii)*$e((m\kappa)_{(m\ell)}^{(1)}) = 1 + \frac{(m\kappa)_{(m\ell)}^{(1)}}{1!} + \frac{(m\kappa)_{(m\ell)}^{(2)}}{2!} + \frac{(m\kappa)_{(m\ell)}^{(3)}}{3!} + \cdots + \infty.$

Lemma 4.2.3. *Let* $\kappa_\ell^{(rn)} \neq 0$, *where* $n \in \mathbb{N}$ *and* $|\ell| < 1$. *Then,*

$$\Delta_\ell e(\kappa_\ell^{(-n)}) = \frac{-n\ell}{(\kappa+\ell)_\ell^{(n+1)}} e((\kappa - n\ell)_\ell^{(-n)}). \tag{4.4}$$

Proof. From (4.3), $e(\kappa_\ell^{(-n)}) = 1 + \frac{1}{1!}\frac{1}{\kappa_\ell^{(n)}} + \frac{1}{2!}\frac{1}{\kappa_\ell^{(2n)}} + \frac{1}{3!}\frac{1}{\kappa_\ell^{(3n)}} + \cdots + \infty$

$$\begin{aligned}
\Delta_\ell(e(\kappa_\ell^{(-n)})) &= \Delta_\ell(1 + \frac{1}{1!}\frac{1}{\kappa_\ell^{(n)}} + \frac{1}{2!}\frac{1}{\kappa_\ell^{(2n)}} + \frac{1}{3!}\frac{1}{\kappa_\ell^{(3n)}} + \cdots + \infty) \\
&= (1-1) + \Delta_\ell \frac{1}{\kappa_\ell^{(n)}} + \Delta_\ell \frac{1}{2!}\frac{1}{\kappa_\ell^{(2n)}} + \Delta_\ell \frac{1}{3!}\frac{1}{\kappa_\ell^{(3n)}} + \cdots \\
&= \frac{1}{1!}\frac{-n\ell}{(\kappa+\ell)_\ell^{(n+1)}} + \frac{1}{2!}\frac{-2n\ell}{(\kappa+\ell)_\ell^{(2n+1)}} + \frac{1}{3!}\frac{-3n\ell}{(\kappa+\ell)_\ell^{(3n+1)}} + \cdots \\
&= \frac{-n\ell}{(\kappa+\ell)_\ell^{(n+1)}} \left(1 + \frac{1}{1!}\frac{1}{(\kappa - n\ell)_\ell^{(n)}} + \frac{1}{2!}\frac{1}{(\kappa - n\ell)_\ell^{(2n)}} + \cdots \right),
\end{aligned}$$

which gives (4.4). □

Lemma 4.2.4. *Let* κ *and* $\ell > 0$. *Then any positive* κ *and* $\ell \in \mathbb{N}$, *we have*

$$e(-\kappa_\ell^{(-1)}) = 1 - \frac{1}{1!}\frac{1}{\kappa_\ell^{(1)}} + \frac{1}{2!}\frac{1}{\kappa_\ell^{(2)}} - \frac{1}{3!}\frac{1}{-\kappa_\ell^{(3)}} + \cdots \infty.$$

The proof follows from the definition of extorial function.

4.3 Higher Order Extorial and its Difference

In this section we define higher order extorial function apply Δ_ℓ on it and obtain some results relevant results.

Definition 4.3.1. *For $\ell \in (-1,1)$ and $\kappa \in \mathbb{R}$, the n^{th} order ℓ-extorial function denoted as $e_n(\kappa_\ell)$ is defined as*

$$e_n(\kappa_\ell) = 1 + \frac{\kappa_\ell^{(n)}}{n!} + \frac{\kappa_\ell^{(2n)}}{(2n)!} + \frac{\kappa_\ell^{(3n)}}{(3n)!} + \cdots + \infty. \tag{4.5}$$

From the definition of extorial function, we obtain following lemma.

Lemma 4.3.2. *For any real κ and $\ell, n \in \mathbb{N}$, we have*

(i)
$$e_n(-\kappa_\ell) = \begin{cases} e_n(\kappa_{(-\ell)}) & \text{if } n \text{ is even} \\ 1 - \frac{\kappa_{(-\ell)}^{(n)}}{n!} + \frac{\kappa_{(-\ell)}^{(2n)}}{(2n)!} - \frac{\kappa_{(-\ell)}^{(3n)}}{(3n)!} + \cdots & \text{if } n \text{ is odd} \end{cases}$$

and

(ii)
$$e_n(-\kappa_{(-\ell)}) = \begin{cases} e_n(\kappa_{(\ell)}) & \text{if } n \text{ is even} \\ 1 - \frac{(\kappa)_\ell^{(n)}}{n!} + \frac{(\kappa)_\ell^{(2n)}}{2n!} - \frac{(\kappa)_\ell^{(3n)}}{3n!} + \cdots & \text{if } n \text{ is odd} \end{cases}$$

Lemma 4.3.3. *Let $\kappa \in \mathbb{R}$ and $n, \ell \in \mathbb{N}$. Then, we have*

$$\Delta_\ell e_n(\kappa_\ell) = \ell \sum_{m=1}^{\infty} \frac{\kappa_\ell^{(mn-1)}}{(mn-1)!}, \ nm \neq 1.$$

Proof. We shall prove this by induction method

$$e_2(\kappa_\ell) = 1 + \frac{\kappa_\ell^{(2)}}{2!} + \frac{\kappa_\ell^{(4)}}{4!} + \frac{\kappa_\ell^{(6)}}{6!} + \cdots + \infty$$
$$\Delta_\ell e_2(\kappa_\ell) = \Delta_\ell \frac{\kappa_\ell^{(2)}}{2!} + \Delta_\ell \frac{\kappa_\ell^{(4)}}{4!} + \Delta_\ell \frac{\kappa_\ell^{(6)}}{6!} + \cdots + \infty = \ell \left[\frac{\kappa_\ell^{(1)}}{1!} + \frac{\kappa_\ell^{(3)}}{3!} + \frac{\kappa_\ell^{(5)}}{5!} + \cdots \right]$$
$$e_3(\kappa_\ell) = 1 + \frac{\kappa_\ell^{(3)}}{3!} + \frac{\kappa_\ell^{(6)}}{6!} + \frac{\kappa_\ell^{(9)}}{9!} + \cdots + \infty$$
$$\Delta_\ell e_3(\kappa_\ell) = \Delta_\ell \frac{\kappa_\ell^{(3)}}{3!} + \Delta_\ell \frac{\kappa_\ell^{(6)}}{6!} + \Delta_\ell \frac{\kappa_\ell^{(9)}}{9!} + \cdots + \infty = \ell \left[\frac{\kappa_\ell^{(2)}}{2!} + \frac{\kappa_\ell^{(5)}}{5!} + \frac{\kappa_\ell^{(8)}}{8!} + \cdots \right]$$

In general, we find for $n \geq 1$

$$\Delta_\ell e_n(\kappa_\ell) = \ell \left[\frac{\kappa_\ell^{(n-1)}}{(n-1)!} + \frac{\kappa_\ell^{(2n-1)}}{(2n-1)!} + \frac{\kappa_\ell^{(3n-1)}}{(3n-1)!} + \cdots \right] = \ell \sum_{m=1}^{\infty} \frac{\kappa_\ell^{(mn-1)}}{(mn-1)!} \qquad \square$$

Lemma 4.3.4. *For any positive integer m, we have $\Delta_\ell^m e_m(\kappa_\ell) = \ell^m e_m(\kappa_\ell)$.*

Proof. $\Delta_\ell e_1(\kappa_\ell) = 0 + \Delta_\ell \frac{\kappa_\ell^{(1)}}{1!} + \Delta_\ell \frac{\kappa_\ell^{(2)}}{2!} + \Delta_\ell \frac{\kappa_\ell^{(3)}}{3!} + \cdots = \ell e_1(\kappa_\ell)$.

$\Delta_\ell e_2(\kappa_\ell) = 0 + \Delta_\ell \frac{\kappa_\ell^{(2)}}{2!} + \Delta_\ell \frac{\kappa_\ell^{(4)}}{4!} + \Delta_\ell \frac{\kappa_\ell^{(6)}}{6!} + \cdots = \frac{2\ell\kappa_\ell(1)}{2!} + \frac{4\ell\kappa_\ell(3)}{4!} + \frac{6\ell\kappa_\ell(5)}{6!} + \cdots$

$\Delta_\ell^2 e_2(\kappa_\ell) = \frac{2\ell(\ell\kappa_\ell^{(0)})}{2!} + \frac{4\ell(3\ell\kappa_\ell^{(2)})}{4!} + \frac{6\ell(5\ell\kappa_\ell^{(4)})}{6!} + \cdots = \ell^2 e_2(\kappa_\ell)$, which yields $\Delta_\ell^m e_m(\kappa_\ell) = \ell^m e_m(\kappa_\ell)$. □

Lemma 4.3.5. *For positive m and real κ, we have $\Delta_\ell^{(-m)} e_m(\kappa_\ell) = \frac{e_m(\kappa_\ell)}{\ell^m}$, $\ell \in \mathbb{N}$.*

Proof. From the lemma (4.3.4), we find $\Delta_\ell^m e_m(\kappa_\ell) = \ell^m e_m(\kappa_\ell)$.

Taking Δ_ℓ^{-m} on both sides, we get

$\Delta_\ell^{-m}\,(\Delta_\ell^m e_m(\kappa_\ell)) = \Delta_\ell^{-m}(\ell^m e_m(\kappa_\ell))$, which gives $\Delta_\ell^{(-m)} e_m(\kappa_\ell) = \frac{e_m(\kappa_\ell)}{\ell^m}$. □

Definition 4.3.6. *For $|\ell| < 1$, and $n \in \mathbb{N}, e_{(-n)}(k_\ell)$ is defined as*

$$e_{(-n)}(\kappa_\ell) = 1 + \frac{1}{n!}\frac{1}{\kappa_\ell^{(n)}} + \frac{1}{(2n)!}\frac{1}{\kappa_\ell^{(2n)}} + \frac{1}{(3n)!}\frac{1}{\kappa_\ell^{(3n)}} + \cdots + \infty. \tag{4.6}$$

Lemma 4.3.7. *For $\ell \in (-1, 1)$ and positive κ, we have*

$$\Delta_\ell e_{(-n)}(\kappa_\ell) = -\ell\Big[\frac{1}{(n-1)!}\frac{1}{(\kappa+\ell)_\ell^{(n+1)}} + \frac{1}{(2n-1)!}\frac{1}{(\kappa+\ell)_\ell^{(2n+1)}} + \frac{1}{(3n-1)!}\frac{1}{(\kappa+\ell)_\ell^{(3n+1)}} + \cdots\Big]$$

Proof. Putting $n = 1$ in (4.6), we get

$e_{(-1)}(\kappa_\ell) = 1 + \frac{1}{1!}\frac{1}{\kappa_\ell^{(1)}} + \frac{1}{2!}\frac{1}{\kappa_\ell^{(2)}} + \frac{1}{3!}\frac{1}{\kappa_\ell^{(3)}} + \cdots + \infty$

$\Delta_\ell e_{(-1)}(\kappa_\ell) = 1 + \Delta_\ell \frac{1}{1!}\frac{1}{\kappa_\ell^{(1)}} + \Delta_\ell \frac{1}{2!}\frac{1}{\kappa_\ell^{(2)}} + \Delta_\ell \frac{1}{3!}\frac{1}{\kappa_\ell^{(3)}} + \cdots + \infty$

$$= -\ell\left[\frac{1}{(\kappa+\ell)_\ell^{(2)}} + \frac{1}{1!}\frac{1}{(\kappa+\ell)_\ell^{(3)}} + \frac{1}{2!}\frac{1}{(\kappa+\ell)_\ell^{(4)}} + \cdots\right].$$

Putting $n = 2$ in (4.6), we get

$$e_{(-2)}(\kappa_\ell) = 1 + \frac{1}{2!}\frac{1}{\kappa_\ell^{(2)}} + \frac{1}{4!}\frac{1}{\kappa_\ell^{(4)}} + \frac{1}{6!}\frac{1}{\kappa_\ell^{(6)}} + \cdots + \infty$$

$$\Delta_\ell e_{(-2)}(\kappa_\ell) = 1 + \Delta_\ell\frac{1}{2!}\frac{1}{\kappa_\ell^{(2)}} + \Delta_\ell\frac{1}{4!}\frac{1}{\kappa_\ell^{(4)}} + \Delta_\ell\frac{1}{6!}\frac{1}{\kappa_\ell^{(6)}} + \cdots + \infty$$

$$= -\ell\left[\frac{1}{1!}\frac{1}{(\kappa+\ell)_\ell^{(3)}} + \frac{1}{3!}\frac{1}{(\kappa+\ell)_\ell^{(5)}} + \frac{1}{5!}\frac{1}{(\kappa+\ell)_\ell^{(7)}} + \cdots\right].$$

Putting $n = 3$ in (4.6), we get

$$e_{(-3)}(\kappa_\ell) = 1 + \frac{1}{3!}\frac{1}{\kappa_\ell^{(3)}} + \frac{1}{6!}\frac{1}{\kappa_\ell^{(6)}} + \frac{1}{9!}\frac{1}{\kappa_\ell^{(9)}} + \cdots + \infty$$

$$\Delta_\ell e_{(-3)}(\kappa_\ell) = 1 + \Delta_\ell\frac{1}{3!}\frac{1}{\kappa_\ell^{(3)}} + \Delta_\ell\frac{1}{6!}\frac{1}{\kappa_\ell^{(6)}} + \Delta_\ell\frac{1}{9!}\frac{1}{\kappa_\ell^{(9)}} + \cdots + \infty$$

$$= -\ell\left[\frac{1}{2!}\frac{1}{(\kappa+\ell)_\ell^{(4)}} + \frac{1}{5!}\frac{1}{(\kappa+\ell)_\ell^{(7)}} + \frac{1}{8!}\frac{1}{(\kappa+\ell)_\ell^{(10)}} + \cdots\right].$$

In general,

$$\Delta_\ell e_{(-n)}(\kappa_\ell) = -\ell\Big[\frac{1}{(n-1)!}\frac{1}{(\kappa+\ell)_\ell^{(n+1)}} + \frac{1}{(2n-1)!}\frac{1}{(\kappa+\ell)_\ell^{(2n+1)}} + \frac{1}{(3n-1)!}\frac{1}{(\kappa+\ell)_\ell^{(3n+1)}} + \cdots\Big].$$

□

4.4 Extorial Type Solution of Difference Equation

In this section, we obtain extorial type solutions of higher order linear $\ell-$difference equations with constant coefficients.

Consider the n^{th} order linear difference equation

$$\left(a_n\frac{\Delta_\ell^n}{\ell^n} + a_{n-1}\frac{\Delta_\ell^{n-1}}{\ell^{n-1}} + \cdots + a_0\right)u(\kappa) = e_1(t\kappa)_{t\ell}, \tag{4.7}$$

where $a_i's$ for $i = 1, 2, 3, ..., n$ are constants. Now we consider the homogenous equation

$$\left(a_n \frac{\Delta_\ell^n}{\ell^n} + a_{n-1}\frac{\Delta_\ell^{n-1}}{\ell^{n-1}} + \cdots + a_0\right) u(\kappa) = 0. \tag{4.8}$$

Assume that $u(\kappa) = e_1((m\kappa)_{(m\ell)}$ as solution of (4.8). Then we get

$$\left(a_n \frac{\Delta_\ell^n e_1((m\kappa)_{(m\ell)}}{\ell^n} + a_{n-1}\frac{\Delta_\ell^{n-1} e_1((m\kappa)_{(m\ell)}}{\ell^{n-1}} + \cdots + a_0 e_1((m\kappa)_{(m\ell)}\right) u(\kappa) = 0. \tag{4.9}$$

Now $\Delta_\ell e_1(m\kappa)_{(m\ell)} = m\ell e_1(m\kappa)_{(m\ell)}$, $\Delta_\ell^2 e_1(m\kappa)_{(m\ell)} = (m\ell)^2 e_1(m\kappa)_{(m\ell)}$.

In general, $\Delta_\ell^n e_1(m\kappa)_{(m\ell)} = (m\ell)^n e_1(m\kappa)_{(m\ell)}$.

Substituting the values in (4.9), we get

$\dfrac{a_n}{\ell^n}(m\ell)^n e_1(m\kappa)_{m\ell} + \dfrac{a_{n_1}}{\ell^{n-1}}(m\ell)^{n-1} e_1(m\kappa)_{m\ell} + \cdots + a_0 e_1(m\kappa)_{m\ell} = 0,$

which gives

$$\left(\frac{a_n}{\ell^n}(m\ell)^n + \frac{a_{n_1}}{\ell^{n-1}}(m\ell)^{n-1} + \cdots + a_0\right) = 0. \tag{4.10}$$

The auxiliary equation for (4.10) is obtained as

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_0 = 0. \tag{4.11}$$

Therefore, suppose that m is a root of (4.11), $e_1(m\kappa)_{(m\ell)}$ is solution of (4.8).

To find particular solution, since

$$\Delta_\ell e_1(t\kappa)_{t\ell} = e_1(t\kappa)_{(t\ell)}(\Delta_\ell e_1(t\kappa)_{(t\ell)} - 1), \Delta_\ell^2 e_1(t\kappa)_{(t\ell)} = e_1(t\kappa)_{(t\ell)}(\Delta_\ell e_1(t\kappa)_{(t\ell)} - 1)^2$$

and in general, $\Delta_\ell^n e_1(t\kappa)_{(t\ell)} = e_1(t\kappa)_{(t\ell)}(\Delta_\ell e_1(t\kappa)_{(t\ell)} - 1)^n$, we get

$\left[a_n \Delta_\ell^n + a_{n-1}\Delta_\ell^{n-1} + \cdots + a_0\right]$

$$\left\{\frac{e_1(t\kappa)_{(t\ell)}}{a_n e_1((t\ell)-1)^n + a_{n-1}e_1((t\ell)-1)^{n-1}+\cdots+a_0}\right\} = e_1(t\kappa)_{(t\ell)}.$$

Hence the particular solution of (4.7) is obtained as

$$u(\kappa) = \frac{e_1(t\kappa)_{(t\ell)}}{a_n e_1((t\ell)-1)^n + a_{n-1}e_1((t\ell)-1)^{n-1}+\cdots+a_0}.$$

Case 1 : Suppose zeros are real and different,then the complementary function for (4.7) is $u(\kappa) = A_1 e_1(m_1\kappa)_{(m_1\ell)} + A_2 e_1(m_2\kappa)_{(m_2\ell)} + \cdots + A_n e_1(m_2\kappa)_{(m_n\ell)}$, where A_i are are constants, for all i=0,1,2,$\cdots n$. Therefore the general solution of (4.7) is

$$\begin{aligned} u(\kappa) = {} & \left[A_1 e_1(m_1\kappa)_{(m_1\ell)} + A_2 e_1(m_2\kappa)_{(m_2\ell)} + \cdots + A_n e_1(m_n\kappa)_{(m\ell)}\right] \\ & + \frac{e_1(t\kappa)_{(t\ell)}}{a_n e_1((t\ell)-1)^n + a_{n-1}e_1((t\ell)-1)^{n-1}+\cdots+a_0}. \end{aligned} \tag{4.12}$$

Case 2 : Suppose the roots are real and same, then the general solution of (4.7) is

$$\begin{aligned} u(\kappa) = {} & \left[A_n + A_{n-1}(m\kappa)^{(n-1)}_{(m\ell)} + A_{n-2}(m\kappa)^{(n-2)}_{(m\ell)} + \cdots + A_1(m\kappa)_(m\ell)^{(1)}\right] e_1(m\kappa)_{(m\ell)} \\ & + \frac{e_1(t\kappa)_{(t\ell)}}{a_n e_1((t\ell)-1)^n + a_{n-1}e_1((t\ell)-1)^{n-1}+\cdots+a_0}. \end{aligned} \tag{4.13}$$

The following example illustration (4.12) and (4.13).

4.5 Example

In this section, we present example to illustrate use of extorial function.

Example 4.5.1. *Consider the linear homogeneous difference equation*

$$\left(\frac{\Delta_\ell^2}{\ell^2} - 4\frac{\Delta\ell}{\ell} + 3\right) u(\kappa) = 0. \tag{4.14}$$

The auxiliary equation is $m^2 - 4m + 3 = (m-1)(m-3) = 0$.

Therefore roots are $m_1 = 1$ and $m_2 = 3$ and (4.14) has a solution.

$$\text{From case 1, } u(\kappa) = Ae_1((\kappa)_\ell) + Be_1((3\kappa)_{(3\ell)}) \tag{4.15}$$

Example 4.5.2. *Consider the difference equation*

$$\left(\frac{\Delta_\ell^2}{\ell^2} - 2\frac{\Delta\ell}{\ell} + 1\right) u(\kappa) = 0. \tag{4.16}$$

The auxilary equation is $m^2 - 2m + 1 = (m-1)(m-1) = 0$. Therefore the roots are m= (1,1) that is real and same. The complementary function is $u(\kappa) = (A + B\kappa)\, e_1(\kappa_\ell) = (A + B\kappa)\, e_1(\kappa_\ell)$.

The following example illustrate 4.12 and 4.13

Example 4.5.3. *Consider the linear non-homogeneous difference equation*

$$\frac{\Delta_\ell^3 u(\kappa)}{\ell^3} - 3\frac{\Delta_\ell^2 u(\kappa)}{\ell^2} + 3\frac{\Delta_\ell u(\kappa)}{\ell} - u(\kappa) = e_1(t\kappa)_{t\ell}. \tag{4.17}$$

The auxilary equation of (4.17) is given by

$$m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0.$$

So roots are $m = (1, 1, 1)$ that is real and equal.

Therefore the general function is $u(\kappa) = \left[A + B(\kappa)_\ell^{(1)} + C(\kappa)_\ell^{(\nu)}\right] e_1(\kappa_\ell)$

$$+ \frac{e_1(t\kappa)_{(t\ell)}}{a_n e_1((t\ell) - 1)^n + a_{n-1} e_1((t\ell) - 1)^{n-1} + \cdots + a_0}.$$

We observe that extorial functions serve as solution of linear higher order

ℓ-delta difference equation. Hence it is posible to arrive at several applications in life sciences.

Chapter 5

Riemann zeta Factorial Function

The extorial function is a sum of ratio of polynomial factorial functions with factorials. This motivates us to introduce Riemann zeta factorial function like extorial function. The zeta factorial function is a sum of reciprocals of factorial polynomials.

5.1 Basic Definitions

In this section, we introduce Riemann zeta factorial function and its fractional order, which is an extension of Riemann zeta function and obtain certain results using difference operator.

Lemma 5.1.1. $1^{(m)} + 2^{(m)} + 3^{(m)} + \cdots + \kappa^{(m)} = \dfrac{(\kappa+1)^{(m+1)}}{(m+1)!}$.

Proof: *Let* $1+2+3+\cdots+\kappa = \dfrac{(\kappa+1)^{(2)}}{(2)!}$

$$\sum_{r=1}^{\kappa}(\kappa-r)^{(2)} = \frac{\kappa^{(3)}}{3!}, \qquad \overset{-1}{\Delta}\,\kappa^{(2)} = \frac{\kappa^{(3)}}{3!}\Big|_0^{\kappa} = \frac{\kappa^{(3)}}{3!} - \frac{0^{(3)}}{3!}$$

$$(\kappa-1)^{(2)} + (\kappa-2)^{(2)} + (\kappa-3)^{(2)} + \cdots + (2)^{(2)} + (1)^{(2)} + (0)^{(2)} = \frac{\kappa^{(3)}}{3!}$$

$$1^{(2)} + 2^{(2)} + \cdots + (\kappa-1)^2 = \frac{\kappa^{(3)}}{3!}$$

Replacing κ *by* $\kappa+1$,

$$1^{(2)} + 2^{(2)} + \cdots + (\kappa)^2 = \frac{(\kappa+1)^{(3)}}{3!}$$

Similarly, we get

$$1^{(m)} + 2^{(m)} + 3^{(m)} + \cdots + \kappa^{(m)} = \frac{(\kappa+1)^{(m+1)}}{(m+1)!}$$

Let $u(\kappa)$ be a real valued function defined on $(-\infty,\infty)$ and $\ell > 0$. The ℓ-difference operator denoted as Δ_ℓ on $u(\kappa)$ is defined by

$$\Delta_\ell u(\kappa) = u(\kappa+\ell) - u(\kappa). \tag{5.1}$$

If there exits a function $v(\kappa)$ such that $\Delta_\ell v(\kappa) = u(\kappa)$, then $v(\kappa)$ is said to be the inverse difference of $u(\kappa)$ and is denoted as $v(\kappa) = \Delta_\ell^{-1} u(\kappa)$.

The polynomial factorial having shift value ℓ is defined by

$$\kappa_\ell^{(n)} = \kappa(\kappa-\ell)(\kappa-2\ell)\cdots(\kappa-(n-1)\ell) \text{ and } \kappa_\ell^{(0)} = 1. \tag{5.2}$$

For $-1 < \ell < 1$ and $\kappa \in (-\infty,\infty)$, we have the extorial function

$$e^{(\kappa)_\ell^{(1)}} = \frac{\kappa_\ell^{(0)}}{0!} + \frac{\kappa_\ell^{(1)}}{1!} + \frac{\kappa_\ell^{(2)}}{2!} + \frac{\kappa_\ell^{(3)}}{3!} + \cdots \tag{5.3}$$

Lemma 5.1.2. *[49] If $\ell > 0$, the inverse principle of Δ_ℓ is given by*

$$\Delta_\ell^{-1} u(\kappa)\Big|_\kappa^\infty = \sum_{r=0}^{\infty} u(\kappa + r\ell). \tag{5.4}$$

If we assume that $\overset{-1}{\underset{\ell}{\Delta}} u(\infty) = \overset{-1}{\underset{\ell}{\Delta}} u(\kappa)\big|_{\kappa=\infty} = c$, then

$$\Delta_\ell^{-1} u(\kappa)\big|_\kappa^\infty = \overset{-1}{\underset{\ell}{\Delta}} u(\infty) - \overset{-1}{\underset{\ell}{\Delta}} u(\kappa)$$
$$= c - \left(-\sum_{r=0}^{\infty} u(\kappa + r\ell)\right) - c = \sum_{r=0}^{\infty} u(\kappa + r\ell).$$

Definition 5.1.3. *For $\ell > 0$, $s \in N(2)$ and $(\kappa + t\ell)_\ell{}^{(s)} \neq 0$, the Riemann zeta factorial function is defined by*

$$\zeta_\ell(\kappa, s) = \sum_{t=0}^{\infty} \frac{1}{(\kappa + t\ell)_\ell{}^{(s)}}. \tag{5.5}$$

Theorem 5.1.4. *Let $s \geq 2$, $\ell \geq 0$ and $(\kappa - 1)_\ell^{(s-1)} \neq 0$. Then, we have*

$$\zeta_\ell(\kappa, s) = \frac{1}{((s-1)\ell(\kappa - \ell))_\ell{}^{(s-1)}} = \sum_{t=0}^{\infty} \frac{1}{(\kappa + t\ell)_\ell{}^{(s)}}.$$

Proof. The infinite series form of $\Delta_\ell^{-1} u(\kappa)$ is given by

$$\Delta_\ell^{-1} u(\kappa) = -\sum_{r=0}^{\infty} u(\kappa + r\ell) + c \tag{5.6}$$

and $\Delta_\ell^{-1} \dfrac{1}{\kappa_\ell^{(s)}} = \dfrac{1}{(s-1)_\ell^{(s-1)}}$+c, where c is a constant. □

5.2 Higher Order Riemann Zeta Factorial Functions

In this section, we define the higher order and fractional order zeta factorial function and derive several identities related to infinite series.

Definition 5.2.1. *For $\nu > 0$, the fractional order Riemann zeta factorial function is defined as*

$$\zeta_\ell^\nu(\kappa, s) = \Delta_\ell^{-(\nu-1)} \zeta_\ell(\kappa, s). \tag{5.7}$$

Theorem 5.2.2. *Let $\nu = 2$, $s \geq 3$, $\ell > 0$ and $(\kappa - 2\ell)_\ell^{s-2} \neq 0$. Then we have the second order Riemann zeta factorial function as*

$$\zeta^2(\kappa, s) = \sum_{t=0}^{\infty} \frac{(t+1)_1^{(1)}}{(\kappa + t\ell)_\ell^{(s)}} = \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} \frac{1}{(\kappa + (r+t)\ell)_\ell^{(s)}} = \frac{1}{\ell^2 (s-1)_1^{(2)} (\kappa - 2\ell)_\ell^{(s-2)}} \tag{5.8}$$

In general, m^{th} order Riemann zeta factorial function is expressed as

$$\zeta_\ell^m(\kappa, s) = \sum_{t=0}^{\infty} \frac{(t + (m-1))_1^{(m-1)}}{(m-1)!(\kappa + t\ell)_\ell^{(s)}} = \frac{1}{\ell^m (s-1)_1^{(m)} (\kappa - m\ell)_\ell^{(s-m)}}. \tag{5.9}$$

Example 5.2.3. *when $m = 3, s = 5, \kappa = 7, \ell = 1$, equation (5.9) becomes*
$\zeta_1^3(7,5) = \sum_{t=0}^{\infty} \frac{(t+1)_1^{(2)}}{(2)!(7+(t))_1^{(5)}} = \frac{1}{(4)_1^{(3)}(4)_1^{(2)}}.$

Remark: *Here, we give first three order zeta factorial functions*

$$\zeta_\ell^1(\kappa,s)=\sum_{t=1}^{\infty}\frac{(t-1)_1^{(0)}}{0!(\kappa+(t-1)\ell)_\ell^{(s)}}=\sum_{t=1}^{\infty}\frac{1}{(\kappa+(t-1)\ell)_\ell^{(s)}}$$

$$\zeta_\ell^2(\kappa,s)=\sum_{t=1}^{\infty}\frac{t_1^{(1)}}{1!(\kappa+(t-1)\ell)_\ell^{(s)}}=\sum_{t=1}^{\infty}\frac{t_1^1}{(\kappa+(t-1)\ell)_\ell^{(s)}}$$

$$\zeta_\ell^3(\kappa,s)=\sum_{t=1}^{\infty}\frac{(t+1)_1^{(2)}}{1.2(\kappa+(t-1)\ell)_\ell^{(s)}}$$

In general,

we can get (5.9), *since* $\sum_{t=0}^{\infty}\frac{1}{(K+t\ell)_\ell^{(s)}}=\sum_{t=1}^{\infty}\frac{1}{(\kappa+(t-1)\ell)_\ell^{(s)}}$

The following corollary gives relation between Riemann Zeta Factorial and Summation.

Corollary 5.2.4. *If* $(\kappa-2\ell)_\ell^{(s-2)}\neq 0$ *and* $s\geq 2+1$, *then we have*

$$\zeta_\ell^3(\kappa,s)-\zeta_\ell^3(\kappa+5\ell,s)-5\zeta_\ell^2(\kappa+2\ell,s)+5\sum_{t=1}^{3}\frac{t}{(\kappa+(t+1)\ell)_\ell^{(s)}}=\sum_{t=0}^{5-1}\frac{(t+2)_1^{(2)}}{2!(\kappa+t\ell)_\ell^{(s)}},$$

$$\zeta_\ell^3(\kappa,s)-\zeta_\ell^3(\kappa+7\ell,s)-7\zeta_\ell^2(\kappa+3\ell,s)+7\sum_{t=1}^{4}\frac{t}{(\kappa+(t+2)\ell)_\ell^{(s)}}=\sum_{t=0}^{7-1}\frac{(t+2)_1^{(2)}}{2!(\kappa+t\ell)_\ell^{(s)}}.$$

and in general,

$$\zeta_\ell^3(\kappa,s)-\zeta_\ell^3(\kappa+(2m+1)\ell,s)-(2m+1)\zeta_\ell^2(\kappa+m\ell,s)+(2m+1)\sum_{t=1}^{m+1}\frac{t}{(\kappa+(t+(m-1))\ell)_\ell^{(s)}}=\sum_{t=0}^{2m}\frac{(t+2)_1^{(2)}}{2!(\kappa+t\ell)_\ell^{(s)}}.$$

$$\zeta_\ell^3(x,s)-\zeta_\ell(\kappa+(2m+1)\ell,s)-(2m+1)\zeta_\ell^{(2)}(\kappa+(2m-1),s)=\sum_{t=0}^{2m}\frac{(t+2)_\ell^{(2)}}{2!(\kappa+t\ell)_\ell^{(s)}}-\frac{(2m+1)(t+1)_\ell^{(1)}}{(\kappa+t\ell)_\ell^{(s)}}.$$

Proof: *The proof follows from* (5.9).

5.3 Finite Summation Formula

Theorem 5.3.1. *Assume that $u(\kappa)$ be a real valued function satisfying the condition $\Delta_\ell^{-1}u(\kappa), \Delta_\ell^{-2}u(\kappa), \Delta_\ell^{-3}u(\kappa), \cdots, \Delta_\ell^{-n}u(\kappa)$ at $\kappa = \infty$ in Z. Then,*

$$\Delta_\ell^{-n}u(\kappa)\Big|_{\kappa+m\ell}^{\kappa} = \sum_{r=0}^{m-1} \frac{(r+(n-1))_1^{(n-1)}}{(n-1)!} u(\kappa+r\ell) + \sum_{t=1}^{n-1} \frac{(m+(n-1-t))_1^{(n-t)}}{(n-t)!} \Delta_\ell^{-t}u(\kappa+m\ell) \tag{5.10}$$

Proof. From (5.6), we obtain

$$\Delta_\ell^{-1}u(\kappa)\Big|_{\kappa}^{\infty} = u(\kappa)+u(\kappa+\ell)+u(\kappa+2\ell)+\cdots+u(\kappa+m\ell)+u(\kappa+(m+1)\ell)+\cdots \tag{5.11}$$

$$\Delta_\ell^{-1}u(\kappa)\Big|_{\kappa+m\ell}^{\infty} = u(\kappa+m\ell) + u(\kappa+(m+1)\ell) + (m+2)\ell) + \cdots \tag{5.12}$$

From (5.11) and (5.12), we get

$$\Delta_\ell^{-1}u(\kappa)\Big|_{\kappa}^{\kappa+m\ell} = \sum_{r=0}^{m-1} u(\kappa+r\ell) \tag{5.13}$$

From (5.11), we arrive

$$\begin{aligned}\Delta_\ell^{-1}\left(\Delta_\ell^{-1}u(\kappa)\right)\Big|_{\kappa}^{\infty} &= \underset{\ell}{\overset{-1}{\Delta}} u(\kappa) + \underset{\ell}{\overset{-1}{\Delta}} u(\kappa+\ell) + \underset{\ell}{\overset{-1}{\Delta}} u(\kappa+2\ell) + \cdots \\ &= u(\kappa) + u(\kappa+\ell) + u(\kappa+2\ell) + \cdots \\ &\quad + u(\kappa+\ell) + u(\kappa+2\ell) + \cdots \\ &\quad + u(\kappa+2\ell) + u(\kappa+3\ell) + \cdots\end{aligned}$$

$$\Delta_\ell^{-2}u(\kappa)\Big|_{\kappa}^{\infty} = u(\kappa)+2u(\kappa+\ell)+3u(\kappa+2\ell)+\cdots+mu(\kappa+(m-1)\ell)+(m+1)u(\kappa+m\ell)+\cdots \tag{5.14}$$

Replacing κ by $\kappa + m\ell$ in the limit of (5.14), we get

$$\Delta_\ell^{-2} u(\kappa)\Big|_{\kappa+m\ell}^{\infty} = u(\kappa+m\ell) + 2u(\kappa+(m+1)\ell) + 3u(\kappa+(m+2)\ell) + \cdots \quad (5.15)$$

(5.14) - (5.15) gives

$$\Delta_\ell^{-2} u(\kappa)\Big|_{\kappa+m\ell}^{\kappa} = \sum_{r=0}^{m-1} \frac{(r+1)_1^{(1)}}{1!} u(\kappa+r\ell) + \frac{m_1^{(1)}}{1!} \underset{\ell}{\overset{-1}{\Delta}} u(\kappa)\Bigg|_{\kappa+m\ell}^{\infty}$$

$$\Delta_\ell^{-1}\left(\Delta_\ell^{-2} u(\kappa)\right)\Big|_{\kappa}^{\infty} = \Delta_\ell^{-1} u(\kappa) + 2\underset{\ell}{\overset{-1}{\Delta}} u(\kappa+\ell) + 3\underset{\ell}{\overset{-1}{\Delta}} u(\kappa+2\ell) + \cdots$$

$$= u(\kappa) + u(\kappa+\ell) + u(\kappa+2\ell) + \cdots$$

$$+2u(\kappa+\ell) + 2u(\kappa+2\ell) + 2u(\kappa+3\ell) + \cdots$$

$$+3u(\kappa+3\ell) + 3u(\kappa+3\ell) + 3u(\kappa+3\ell) + \cdots$$

$$\Delta_\ell^{-3} u(\kappa)\Big|_{\kappa}^{\infty} = u(\kappa) + (1+2)u(\kappa+\ell) + (1+2+3)u(\kappa+2\ell) + (1+2+3+4)u(\kappa+3\ell) + \cdots$$

$$+(1+2+\cdots+(m+1))u(\kappa+m\ell)$$

Replacing κ by $\kappa + m\ell$ in the limits

$$\Delta_\ell^{-3} u(\kappa)\Big|_{\kappa+m\ell}^{\infty} = u(\kappa+m\ell) + (1+2)u(\kappa+(m+1)\ell) + (1+2+3)u(\kappa+(m+2)\ell) + \cdots \quad (5.16)$$

$$\Delta_\ell^{-3} u(\kappa)\Big|_{\kappa}^{\infty} = u(\kappa) + (1+2)u(\kappa+\ell) + (1+2+3)u(\kappa+2\ell) + (1+2+3+4)u(\kappa+3\ell) + \cdots$$

$$+(1+2+\cdots+m)u(\kappa+(m-1)\ell) + (1+2+\cdots+(m+1))u(\kappa+m\ell)$$

$$+(1+2+\cdots+(m+2))u(\kappa+(m+1)\ell) + \cdots$$

Applying the formula $1+2+\cdots+m = (m+1)_2^{(2)}$

$$\Delta_\ell^{-3} u(\kappa)\Big|_{\kappa}^{\infty} = u(\kappa) + \frac{3_1^{(2)}}{2!} u(\kappa+\ell) + \frac{4_1^{(2)}}{2!} u(\kappa+2\ell) + \cdots + \frac{(m+1)_1^{(2)}}{2!} u(\kappa+(m-1)\ell)$$

$$+(1+2+\cdots+(m+1))u(\kappa+m\ell)+(1+2+\cdots+(m+2))u(\kappa+(m+1)\ell)$$

$$+(1+2+\cdots+(m+3))u(\kappa+(m+2)\ell)+\cdots \tag{5.17}$$

Substracte (5.16) from (5.17)

$$\Delta_\ell^{-3}u(\kappa)\Big|_\kappa^\infty-\Delta_\ell^{-3}u(\kappa)\Big|_{\kappa+m\ell}^\infty=u(\kappa)+\frac{3_1^{(2)}}{2!}u(\kappa+\ell)+\frac{4_1^{(2)}}{2!}u(\kappa+2\ell)+\cdots+$$

$$+\frac{(m+1)_1^{(2)}}{2!}u(\kappa+(m-1)\ell)$$

$$+(1+2+\cdots+m)u(\kappa+m\ell)+mu(\kappa+m\ell)$$

$$+(1+2+\cdots+(m+2))u(\kappa+(m+1)\ell)$$

$$+2mu(\kappa+(m+1)\,\ell)+(1+2+\cdots+(m+2)\,\ell)u(\kappa+(m+2)\ell)$$

$$\Delta_\ell^{-3}u(\kappa)\Big|_{\kappa+m\ell}^{\kappa}=\sum_{r=0}^{m-1}\frac{(r+2)}{2!}u(\kappa+r\ell)+\frac{(m+1)_1^{(2)}}{2!}\mathop{\Delta}_{\ell}^{-1}u(\kappa)\Big|_{\kappa+m\ell}^{\infty}+\frac{m_1^{(1)}}{1!}\Delta_\ell^{-2}u(\kappa)\Big|_{\kappa+m\ell}^{\infty}$$

By induction on n we get the proof of (5.10).

The following example illustrates the Thorem 5.3.1 □

Example 5.3.2. *Taking $n=4$ in (5.10) we arrive at*

$$\Delta_\ell^{-4}u(\kappa)\Big|_\kappa^{\kappa+m\ell}=\sum_{r=0}^{m-1}\tfrac{(r+3)_1^{(3)}}{3!}u(\kappa+r\ell)+\sum_{t=1}^{3}\tfrac{(m+3-t)_1^{(4-t)}}{(3-t)}\Delta_\ell^{-t}u(\kappa+m\ell).$$

Now taking $u(\kappa)=\dfrac{1}{\kappa_\ell^{(s)}}$ and using (5.9), we find

$$\zeta_\ell^4(\kappa,s)-\zeta_\ell^4(\kappa+m\ell,s)=\sum_{r=0}^{m-1}\tfrac{(r+3)_1^{(3)}}{3!}\tfrac{1}{(\kappa+r\ell)_\ell^s}$$

$$+\frac{(m+2)_1^{(3)}}{3!}\zeta_\ell^1(\kappa+m\ell,s)+\frac{(m+1)_1^{(2)}}{2!}\zeta_\ell^2(\kappa+m\ell,s)+\frac{m_1^{(1)}}{1!}\zeta_\ell^3(\kappa+m\ell,s) \tag{5.18}$$

when $\ell = 1$, s=5, $\kappa = 5$, m=3, $\zeta_\ell^4(\kappa, s) = \dfrac{1}{\ell^4 (s-1)_1^4 (\kappa - 4\ell)_\ell^{(s-4)}}$

and $\zeta_\ell^4(\kappa + m\ell, s) = \dfrac{1}{\ell^4 (s-1)_1^4 (\kappa - \ell)_\ell^{(s-4)}}$

First we find LHS of (5.18).

$$LHS = \frac{1}{(4)_1^{(4)}\,(1)_1^{(1)}} - \frac{1}{(4)_1^{(4)}\,(4)_1^{(1)}} = \frac{1}{4\cdot 3\cdot 2\cdot 1} - \frac{1}{4\cdot 3\cdot 2\cdot 1\cdot 4} = \frac{1}{24}\left[1 - \frac{1}{4}\right] = \frac{1}{24}\left[\frac{3}{4}\right] = \frac{1}{32}$$

$$\begin{aligned} RHS = &\sum_{r=0}^{2} \frac{(r+3)_1^{(3)}}{3!} \frac{1}{(5+r)_1^{(5)}} + \frac{5_1^{(3)}}{3!} \frac{1}{1\cdot (4)_1^{(1)}\,(7)_1^{(4)}} \\ &+ \frac{4_1^{(2)}}{2!} \frac{1}{(4)_1^{(2)}\,(6)_1^{(3)}} + \frac{3_1^{(2)}}{2!} \frac{1}{(4)_1^{(3)}\,(5)_1^{(2)}} \\ = &\frac{3_1^{(3)}}{3!} \frac{1}{(5)_1^{(5)}} + \frac{4_1^{(3)}}{3!} \frac{1}{(6)_1^{(5)}} + \frac{5_1^{(3)}}{3!} \frac{1}{(7)_1^{(5)}} + \frac{5_1^{(3)}}{3!} \frac{1}{1\cdot (4)_1^{(1)}\,(7)_1^{(4)}} \\ &+ \frac{4_1^{(2)}}{2!} \frac{1}{(4)_1^{(2)}\,(6)_1^{(3)}} + \frac{3_1^{(2)}}{2!} \frac{1}{(4)_1^{(3)}\,(5)_1^{(2)}} \\ = &\frac{1}{120} + \frac{1}{180} + \frac{1}{252} + \frac{1}{336} + \frac{1}{240} + \frac{1}{160} = \frac{1}{32} \end{aligned}$$

Similarly if we take n=5, we have

$$\Delta_\ell^{-5} u(\kappa)\Big|_{\kappa+m\ell}^{\kappa} = \sum_{r=0}^{m-1} \frac{(r+4)_1^{(3)}}{4!} u(\kappa + r\ell) + \sum_{t=1}^{4} \frac{(m+4-t)_1^{(5-t)}}{(5-t)} \Delta_\ell^{-t} u(\kappa + m\ell).$$

$$\zeta_\ell^5(\kappa, s) - \zeta_\ell^5(\kappa + m\ell, s) = \sum_{r=0}^{m-1} \frac{(r+4)_1^{(3)}}{4!} \frac{1}{(\kappa + r\ell)_\ell^s} + \frac{(m+3)_1^{(4)}}{3!} \zeta_\ell^1(\kappa + m\ell, s)$$

$$+ \frac{(m+2)_1^{(2)}}{3!} \zeta_\ell^2(\kappa + m\ell, s) + \frac{(m+1)_1^{(2)}}{2!} \zeta_\ell^3(\kappa + m\ell, s) + \frac{(m)_1^{(1)}}{1!} \zeta_\ell^4(\kappa + m\ell, s) \quad (5.19)$$

For verification we take $\ell = 1$, $s = 6$, $\kappa = 6$, $m = 2$ *in* (5.19)

Now, LHS is

$$\zeta_\ell^5(\kappa,s)-\zeta_\ell^5(\kappa+m\ell,s)=\frac{1}{(5)_1^{(5)}(1)_1^{(1)}}-\frac{1}{(5)_1^{(5)}(3)_1^{(1)}}=\frac{1}{(5)_1^{(5)}}\left[1-\frac{1}{3}\right]$$
$$=\frac{1}{5\cdot4\cdot3\cdot2\cdot1}\left[\frac{2}{3}\right]=\frac{1}{180}$$

$$RHS=\sum_{r=0}^{1}\frac{(r+4)_1^{(4)}}{4!}\frac{1}{(6+r)_1^{(6)}}+\frac{5_1^{(4)}}{4!}\frac{1}{(5)_1^{(1)}(7)_1^{(5)}}+\frac{4_1^{(3)}}{3!}\frac{1}{(5)_1^{(2)}(6)_1^{(4)}}$$
$$+\frac{3_1^{(2)}}{2!}\frac{1}{(5)_1^{(3)}(5)_1^{(3)}}+\frac{2_1^{(2)}}{1!}\frac{1}{(5)_1^{(3)}(4)_1^{(2)}}$$
$$=\frac{4_1^{(4)}}{4!}\frac{1}{(6)_1^{(6)}}+\frac{5_1^{(4)}}{4!}\frac{1}{(7)_1^{(6)}}+\frac{5_1^{(4)}}{4!}\frac{1}{(5)_1^{(1)}(7)_1^{(5)}}+\frac{4_1^{(3)}}{3!}\frac{1}{(5)_1^{(2)}(6)_1^{(4)}}$$
$$+\frac{3_1^{(2)}}{2!}\frac{1}{(5)_1^{(3)}(5)_1^{(3)}}+\frac{2_1^{(2)}}{1!}\frac{1}{(5)_1^{(3)}(4)_1^{(2)}}$$
$$=\frac{1}{720}+\frac{1}{1008}+\frac{1}{2520}+\frac{1}{1800}+\frac{1}{1200}+\frac{1}{720}=\frac{1}{180}$$

Thus we have verified the Theorem (5.3.1).

5.4 Infinite Summation Formula

Theorem 5.4.1. *(Infinite Summation Formula) Let u be a real valued function ddefined on R and n be a positive integer.*

If $\Delta_\ell^{-1}u(\kappa)\big|_{\kappa=\infty}$, $\Delta_\ell^{-2}u(\kappa)\big|_{\kappa=\infty}$, $\cdots$, $\Delta_\ell^{-1}u(\kappa)\big|_{\kappa=\infty}$, then

$$\overset{-n}{\underset{\ell}{\Delta}}u(\kappa)\Big|_{\kappa}^{\infty}=\sum_{t=0}^{\infty}\frac{(r+(n-1))_1^{(n-1)}}{(n-1)!}u(\kappa+t\ell) \tag{5.20}$$

Proof. From (5.6), we obtain

$$\Delta_\ell^{-1}u(\kappa)\Big|_\kappa^\infty = u(\kappa)+u(\kappa+\ell)+u(\kappa+2\ell)+u(\kappa+3\ell)+\cdots$$

$$\Delta_\ell^{-1}\left(\Delta_\ell^{-1}u(\kappa)\right)\Big|_\kappa^\infty = \overset{-1}{\underset{\ell}{\Delta}}u(\kappa)+\overset{-1}{\underset{\ell}{\Delta}}u(\kappa+\ell)+\overset{-1}{\underset{\ell}{\Delta}}u(\kappa+2\ell)+\cdots$$

$$=u(\kappa)+u(\kappa+\ell)+u(\kappa+2\ell)+\cdots+u(\kappa+\ell)+u(\kappa+2\ell)+u(\kappa+3\ell)+\cdots$$

$$+u(\kappa+2\ell)+u(\kappa+3\ell)+\cdots+u(\kappa+3\ell)+u(\kappa+4\ell)+\cdots$$

$$\Delta_\ell^{-2}u(\kappa)\Big|_\kappa^\infty = u(\kappa)+2u(\kappa+\ell)+3u(\kappa+2\ell)+\cdots$$

$$\Delta_\ell^{-1}\left(\Delta_\ell^{-1}u(\kappa)\right)\Big|_\kappa^\infty = u\overset{-1}{\underset{\ell}{\Delta}}(\kappa)+2\overset{-1}{\underset{\ell}{\Delta}}u(\kappa+\ell)+3\overset{-1}{\underset{\ell}{\Delta}}u(\kappa+2\ell)+\cdots$$

$$=u(\kappa)+u(\kappa+\ell)+u(\kappa+2\ell)+\cdots+2u(\kappa+\ell)+2u(\kappa+2\ell)+2u(\kappa+3\ell)+\cdots$$

$$+3u(\kappa+2\ell)+3u(\kappa+3\ell)+\cdots+4u(\kappa+3\ell)+4u(\kappa+4\ell)+\cdots$$

$$=u(\kappa)+(1+2)u(\kappa+\ell)+(1+2+3)u(\kappa+2\ell)+\cdots$$

$$\Delta_\ell^{-3}u(\kappa)\Big|_\kappa^\infty = u(\kappa)+\frac{(3)_1^{(2)}}{2!}u(\kappa+\ell)+\frac{(4)_1^{(2)}}{2!}u(\kappa+2\ell)+\frac{(5)_1^{(2)}}{2!}u(\kappa+3\ell)+\cdots$$

By induction on n we get the proof of (5.20). □

Corollary 5.4.2. *For $\nu>0$, the γ^{th} order inverse of u*

$$\Delta_\ell^{-\nu}u(\kappa)\Big|_\kappa^\infty = \sum_{t=0}^{\infty}\frac{(t+(\nu-1))_1^{(\nu-1)}}{\lceil\gamma}u(\kappa+t\ell) = \sum_{t=0}^{\infty}\frac{\lceil t+\nu}{\lceil\nu\,\lceil t+1}u(\kappa+t\ell).$$

Proof. The proof follows by replacing n by ν in theorem 5.4.1 and $\kappa_\ell^{(n)} = \frac{\lceil\kappa+1}{\lceil\kappa+1-n}$ □

Example 5.4.3. *The zeta factorial function for fractional order obtained as*

$$\zeta_1^{(\nu)}(\kappa,s)=\sum_{t=0}^{\infty}\frac{(t+(\nu-1))_1^{(\nu-1)}}{\left|\overline{\nu}\right.(\kappa+t)^{(s)}}=\frac{1}{(s-1)^{(\nu)}(\kappa-\nu)_\ell^{(s-\nu)}}$$

$$\zeta_1^{(\nu)}(\kappa,s)=\sum_{t=0}^{\infty}\frac{\left|\overline{t+\nu}\right.}{\left|\overline{\nu}\right.\left|\overline{t+1}\right.}\cdot\frac{\left|\overline{\kappa+t+1-s}\right.}{\left|\overline{\kappa+t+1}\right.}$$

$$=\frac{\left|\overline{s-\nu}\right.}{\left|\overline{s}\right.}\frac{\left|\overline{\kappa-\nu+1-(s-\nu)}\right.}{\left|\overline{\kappa-\nu+1}\right.}$$

$$\zeta_1^{(\nu)}(\kappa,s)=\sum_{t=0}^{\infty}\frac{\left|\overline{t+\nu}\right.}{\left|\overline{\nu}\right.\left|\overline{t+1}\right.}\cdot\frac{\left|\overline{\kappa+t+1-s}\right.}{\left|\overline{\kappa+t+1}\right.}=\frac{\left|\overline{s-\nu}\right.}{\left|\overline{s}\right.}\frac{\left|\overline{\kappa-s+1}\right.}{\left|\overline{\kappa-\nu+1}\right.}$$

If we take $\nu=0.1\quad s=0.5,\quad \kappa=3,$ *then we find*

$$\zeta_1^{(0.1)}(3,0.5)=\sum_{t=0}^{\infty}\frac{\left|\overline{t+0.1}\right.}{\left|\overline{0.1}\right.\left|\overline{t+1}\right.}\cdot\frac{\left|\overline{4+t-0.5}\right.}{\left|\overline{4+t}\right.}=\frac{\left|\overline{0.4}\right.}{\left|\overline{0.5}\right.}\frac{\left|\overline{0.5}\right.}{\left|\overline{3.9}\right.}$$

and hence

$$\frac{\left|\overline{0.1}\right.}{\left|\overline{0.1}\right.\left|\overline{1}\right.}\frac{\left|\overline{3.5}\right.}{\left|\overline{4}\right.}+\frac{\left|\overline{1.1}\right.}{\left|\overline{0.1}\right.\left|\overline{2}\right.}\frac{\left|\overline{4.5}\right.}{\left|\overline{5}\right.}+\frac{\left|\overline{2.1}\right.}{\left|\overline{0.1}\right.\left|\overline{3}\right.}\frac{\left|\overline{5.5}\right.}{\left|\overline{4}\right.}+\cdots=\frac{\left|\overline{0.4}\right.}{\left|\overline{0.5}\right.}\frac{\left|\overline{0.5}\right.}{\left|\overline{3.9}\right.}$$

It is also possible to arrrive

$$\Delta_\ell^{-\nu}e^{-s\kappa}\Big|_\kappa^\infty=\frac{1}{\left(1-e^{-s\ell}\right)^\nu}e^{-s\kappa}$$

and from Corollary 5.4.2 we can obtain

$$\frac{1}{\left(1-e^{-s\ell}\right)^\nu}e^{-s\kappa}\Bigg|_\kappa^\infty=\sum_{t=0}^{\infty}\frac{\left|\overline{t+\nu}\right.}{\left|\overline{\nu}\right.(\kappa+t)^s}e^{-s(\kappa+t\ell)}.$$

Similarly one can obtain several results on zeta factorial functions.

The above are all extorial family functions arrived from factorial function and exponential function.

Chapter 6

Extra, Partial Exponential and Extorial Functions

In [1], the theory of difference equations is developed with the definition of the difference operator $\Delta u(\kappa) = u(\kappa+1) - u(\kappa), \kappa \in N, \mathbb{N}(0) = \{0, 1, 2, \cdots\}$. Similarly one can define $\Delta_\ell u(\kappa) = u(\kappa+\ell) - u(\kappa)$, Where $0 \neq \ell \in R$ and $u : R \to R$ is real valued function. In [27], the authors used the generalized difference operator Δ_ℓ and established basic properties of Δ_ℓ such as product and quotient rules of Δ_ℓ.

6.1 Basic Definitions

The second, third kind and n^{th} kinds of the generalized difference operators are denoted by $\Delta_{\ell,m}, \Delta_{\ell_1,\ell_2,\ell_3}$ and $\Delta_{\ell_1,\ell_2,\ell_3\cdots,\ell_n}$ respectively and defined on $u(\kappa)$ as $\Delta_{\ell,m}u(\kappa) = \Delta_\ell\left(\Delta_m u(\kappa)\right)$, $\Delta_{\ell_1,\ell_2,\ell_3}u(\kappa) = \Delta_{\ell_1}\left(\Delta_{\ell_2}\left(\Delta_{\ell_3}u(\kappa)\right)\right)$
and in general $\Delta_{\ell_1,\ell_2,\ell_3,\cdots,\ell_n}u(\kappa) = \Delta_{\ell_1}\left(\Delta_{\ell_2}\cdots\left(\Delta_{\ell_n}u(\kappa)\right)\right)$

By these operator, we obtain the generalized version of Leibniz theorem, Binomial theorem. The theory of difference operator Δ_ℓ developed in [29] agrees when $\ell = 1$. In the continuous case, we have the integration by parts , Bernoulli's formula and several results in calculus. Motivated by the above situation, we define extra exponential functions, sub exponential functions and extorial functions to obtain solution of certain type differential and difference equations.

6.2 Partial Exponential Function in Differential Equation

In this section, we define sub exponential function, denoted as $e_n(x)$, extra exponential function and find solutions of higher order differential equations.

Definition 6.2.1. *For each positive integer n and for $x \in (-\infty, \infty)$, the partial* *exponential function denoted as $e_n(x)$ is defined as*

$$e_n(x) = 1 + \frac{x^n}{n!} + \frac{x^{2n}}{(2n)!} + \frac{x^{3n}}{(3n)!} + \cdots + \infty = \sum_{r=0}^{\infty} \frac{x^{rn}}{(rn)!} \tag{6.1}$$

when $n \leq 2$, (6.1) becomes partial exponential function.

Theorem 6.2.2. *The partial exponential function $e_n(x)$ is a solution of the $(n-1)^{th}$ order linear nonhomogeneous differential equation*

$$\frac{d^{n-1}}{dx^{n-1}}u(x) + \frac{d^{n-2}}{dx^{n-2}}u(x) + \cdots + \frac{d}{dx}u(x) + u(x) = e^x$$

Proof: From the equation (6.1), we have

$$e_n(x) = 1 + \frac{x^n}{n!} + \frac{x^{2n}}{(2n)!} + \frac{x^{3n}}{(3n)!} + \cdots + \infty.$$

Differentiating with respect to x, we get

$$\frac{d}{dx}e_n(x) = \frac{x^{n-1}}{(n-1)!} + \frac{x^{2n-1}}{(2n-1)!} + \frac{x^{3n-1}}{(3n-1)!} + \cdots + \infty,$$

$$\frac{d^2}{dx^2}e_n(x) = \frac{x^{n-2}}{(n-2)!} + \frac{x^{2n-2}}{(2n-2)!} + \frac{x^{3n-2}}{(3n-2)!} \cdots + \infty,$$

and in general, we find

$$\frac{d^r}{dx^r}e_n(x) = \frac{x^{n-r}}{(n-r)!} + \frac{x^{2n-r}}{(2n-r)!} + \frac{x^{3n-r}}{(3n-r)!} \cdots + \infty$$

Adding above equation for $r = 0, 1, 2, 3, \cdots, n-1$, we get
$\sum_{r=0}^{n-1} \frac{d^r}{dx^r}e_n(x) = e^x$ *, which yields the proof by taking $u(x) = e_n(x)$.*

The definition (6.2.1) can be extended to $\nu > 0$.

Definition 6.2.3. *Let $\nu > 0$ be a any real number and $x \in \mathbb{R}$. Then, the extra exponential function denoted as $e_\nu(x)$ is defined as*

$$e_\nu(x) = 1 + \frac{x^\nu}{\Gamma(\nu+1)} + \frac{x^{2\nu}}{\Gamma(2\nu+1)} + \cdots + \infty = \sum_{r=0}^{\infty} \frac{x^{r\nu}}{\Gamma(r\nu+1)} \tag{6.2}$$

when ν takes the value $0 < \nu < 1$, $\nu = 1$ (6.2) becomes extra exponential, exponential and partial exponential functions respectively.

Remark 6.2.4. *Taking $\kappa = 0$ in (5.4) and then taking $\ell = 1$*

$$\Delta_{-\ell}^{-1} u(\kappa)\Big|_{t=0} = \sum_{r=0}^{\infty} u(r\ell), \quad \Delta_{-1}^{-1} u(\kappa)\Big|_{t=0} = \sum_{r=0}^{\infty} u(r). \tag{6.3}$$

Definition 6.2.5. *[42] Let $\ell > 0$, $\nu \in (-\infty, \infty)$ and $\Gamma(\frac{\kappa}{\ell}+1)$ be the gamma function. Then, the ℓ - factorial polynomial in κ for real index ν is defined by*

$$\kappa_\ell^{(\nu)} = \ell^\nu \frac{\Gamma(\kappa/\ell+1)}{\Gamma(\kappa/\ell+1-\nu)}, \quad \kappa/\ell+1, (\kappa/\ell+1-\nu) \notin -N(0) = \{0, -1, -2, \cdots\}. \tag{6.4}$$

Special Cases: *We give some special case by taking $\ell = \pm 1$.*

(i) *when* $\ell = -1$, $\kappa_{-1}^{(n)} = \kappa(\kappa+1)(\kappa+2)\cdots(\kappa+n-1) = \prod_{r=1}^{n}(\kappa+r-1), n \in \mathbb{Z}^+$.

(ii) *when* $\ell = 1$, $\kappa_1^{(n)} = \kappa(\kappa-1)(\kappa-2)\cdots(\kappa-n+1) = \prod_{r=1}^{n}(\kappa-r+1) = \kappa^{(n)}, n \in \mathbb{Z}^+$.

Theorem 6.2.6. *The relation between extra exponential and exponential for $\nu = 0.5$ is given by*

$$e_{0.5}(x) = \frac{x^{0.5}}{0.5\Gamma 0.5} \left[\Delta_{-2}^{-1} \frac{(2x)^\kappa}{3_{-2}^{(\kappa)}} \,|_{\kappa=0} \right] + e^x. \tag{6.5}$$

Proof: From the definition of extra exponential function for $\nu = 0.5$, we have

$$e_{0.5}(x) = 1 + \frac{x^{0.5}}{\Gamma 1.5} + \frac{x^1}{\Gamma 2} + \frac{x^{1.5}}{\Gamma 2.5} + \frac{x^2}{\Gamma 2} + \frac{x^{2.5}}{\Gamma 2.5} + \cdots, \quad x > 0$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \infty + \frac{x^{0.5}}{0.5\Gamma 0.5} + \frac{x^{1.5}}{(1.5)(0.5)\Gamma 0.5} + \frac{x^{2.5}}{(2.5)(1.5)(0.5)\Gamma 0.5} + \infty$$

$$= \frac{x^{0.5}}{0.5\Gamma 0.5}\left[1 + \frac{x}{(1.5)^{(1)}} + \frac{x^2}{(2.5)^{(2)}} + \frac{x^3}{(3.5)^{(3)}} + \cdots\right] + e^x$$

$$= \frac{x^{0.5}}{0.5\Gamma 0.5}\left[1 + \frac{x}{\frac{3}{2}} + \frac{x^2}{\frac{3}{2}\frac{5}{2}} + \frac{x^3}{\frac{3}{2}\frac{5}{2}\frac{7}{2}} + \cdots\right] + e^x$$

$$= \frac{x^{0.5}}{0.5\Gamma 0.5}\left[1 + \frac{2x}{3.1} + \frac{(2x)^2}{3.5} + \frac{(3x)^3}{3.5.7} + \cdots\right] + e^x$$

$$= \frac{x^{0.5}}{0.5\Gamma 0.5}\left[1 + \frac{2x}{3^{(1)}_{-2}} + \frac{(2x)^2}{3^{(2)}_{-2}} + \frac{(3x)^3}{3^{(3)}_{-2}} + \cdots\right] + e^x.$$

By applying $\Delta^{-1}_{-\ell} u(\kappa) = u(\kappa) + u(\kappa+\ell) + u(\kappa+2\ell) + \cdots$ for $\ell = 1$, we get

$$\Delta^{-1}_{-1} u(\kappa) = u(\kappa) + u(\kappa+1) + u(\kappa+2) + \cdots \tag{6.6}$$

Now, the proof follows by taking $u(\kappa) = \dfrac{x^\kappa}{1^{(\kappa)}_{-1}}$ in (6.6) and then putting $\kappa = 0$.

Corollary 6.2.7. *For $x > 0$, the difference of extra exponential and exponential for $\nu = 0.5$ is given by*

$$\frac{\Gamma 1.5}{x^{0.5}}\left[e_{0.5}(x) - e_1(x)\right] = \overset{-1}{\underset{-1}{\Delta}} \frac{(2x)^\kappa}{3^{(\kappa)}_{-2}} \Big|_{\kappa=0} \tag{6.7}$$

Proof: The proof follows from (6.5).

Corollary 6.2.8. *By taking x=1, we have $\Gamma 1.5\,(e_{0.5}(1) - e) = \Delta^{-1}_{-2} \dfrac{(2x)^\kappa}{3^{(\kappa)}_{-2}} \Big|_{\kappa=0}$.*

Theorem 6.2.9. *If $m \in \mathbb{N}$, we have*

$$e_1(x) - e_m(x) = \sum_{r=1}^{m-1} \Delta^{-1}_{-m} \frac{x^{\kappa+r}}{(\kappa+r)!} \Big|_{\kappa=0}. \tag{6.8}$$

Proof: From the definition of exponential function, we have

$$e_1(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \infty$$

Rearranging the terms into four groups, we get

$$e_1(x) = 1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \frac{x^{12}}{12!} + \cdots + \infty + \frac{x}{1!} + \frac{x^5}{5!} + \frac{x^9}{9!} + \cdots + \infty$$
$$+ \frac{x^2}{2!} + \frac{x^6}{6!} + \frac{x^{10}}{10!} + \cdots + \infty + \frac{x^3}{3!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \cdots + \infty$$

which is same as

$$e_1(x) = e_4(x) + \tfrac{x}{1!}\left[1 + \frac{x^4}{2.3.4.5} + \frac{x^8}{2.3.4.\cdots 9}\right] + \tfrac{x^2}{2!}\left[1 + \frac{x^4}{3.4.5.6} + \frac{x^8}{3.4.5\cdots 10}\right]$$
$$+ \tfrac{x^3}{3!}\left[1 + \frac{x^4}{4.5.6.7} + \frac{x^8}{4.5.6.7.8.9.10 + \cdots}\right]$$

By applying the formula for $\underset{-\ell}{\Delta}^{-1}$*, we get*

$$e_1(x) = e_4(x) + \tfrac{x}{1!}\underset{-4}{\Delta}^{-1}\frac{x^\kappa}{2^{(\kappa)}_{-1}}\ |_{\kappa=0} + \tfrac{x^2}{2!}\underset{-4}{\Delta}^{-1}\frac{x^\kappa}{3^{(\kappa)}_{-1}}\ |_{\kappa=0} + \tfrac{x^3}{3!}\underset{-4}{\Delta}^{-1}\frac{x^\kappa}{4^{(\kappa)}_{-1}}\ |_{\kappa=0}$$

and which is same as

$$e_1(x) - e_4(x) = \sum_{r=1}^{3} \tfrac{x^r}{r!}\underset{-4}{\Delta}^{-1}\frac{x^\kappa}{(r+1)^{(\kappa)}_{-1}}\ |_{\kappa=0} = \sum_{r=1}^{3} \underset{-4}{\Delta}^{-1}\frac{x^{\kappa+r}}{(r+\kappa)!}\ |_{\kappa=0}$$

Continuing this process, we get (6.8) by replacing 4 by $m \in \mathbb{N}$*.*

Theorem 6.2.10. *If* $\nu m = 1, m \in N$*, then we have*

$$e_\nu(x) - e_1(x) = \sum_{r=1}^{m-1} \frac{x^{(0.25)r}}{\Gamma(1 + r(0.25))}\underset{-1}{\Delta}^{-1}\frac{x^\kappa}{(1+\nu r)^{\kappa}_{-1}}\ |_{\kappa=0}\ . \tag{6.9}$$

Proof: From the definition of extra exponential function, by putting $\nu = 0.25$,

$$e_{0.25}(x) = 1 + \frac{x^{0.25}}{\Gamma 0.25 + 1} + \frac{x^{0.5}}{\Gamma 0.5 + 1} + \frac{x^{0.75}}{\Gamma 0.75 + 1} + \frac{x^1}{\Gamma 1 + 1} + \cdots + \infty$$

By rearranging the terms, we find

$$e_{0.25}(x) = \frac{x^{0.25}}{\Gamma 1.25} + \frac{x^{1.25}}{\Gamma 2.25} + \frac{x^{2.25}}{\Gamma 3.25} + \cdots + \infty + \frac{x^{0.5}}{\Gamma 1.5} + \frac{x^{1.5}}{\Gamma 2.5} + \frac{x^{2.5}}{\Gamma 3.5} + \cdots + \infty$$

$$+\frac{x^{0.75}}{\Gamma 1.75}+\frac{x^{1.75}}{\Gamma 2.75}+\frac{x^{2.75}}{\Gamma 3.75}+\cdots+\infty+\frac{x^{0}}{\Gamma 1}+\frac{x^{1}}{\Gamma 2}+\frac{x^{2}}{\Gamma 3}+\cdots+\infty$$

$$=\frac{x^{0.25}}{\Gamma 1.25}\left[1+\frac{x}{(1.25)}+\frac{x^2}{(1.25)(2.25)}+\cdots\right]+\frac{x^{0.5}}{\Gamma 1.5}\left[1+\frac{x}{(1.5)}+\frac{x^2}{(1.5)(2.5)}+\cdots\right]+$$

$$\frac{x^{0.75}}{\Gamma 1.75}\left[1+\frac{x}{(1.75)}+\frac{x^2}{(1.75)(2.75)}+\cdots\right]+\frac{x^{0}}{\Gamma 1}\left[1+\frac{x}{(1)}+\frac{x^2}{(1)(2)}+\frac{x^3}{(1)(2)(3)}+\cdots\right]$$

By applying $\underset{-1}{\Delta^{-1}}u(\kappa)$ *on each group, we get*

$$e_{0.25}(x)=\frac{x^{0.25}}{\Gamma 1.25}\Delta_{-1}^{-1}\frac{x^{\kappa}}{(1.25)_{-1}^{(\kappa)}}\ |_{\kappa=0}+\frac{x^{0.5}}{\Gamma 1.5}\Delta_{-1}^{-1}\frac{x^{\kappa}}{(1.5)_{-1}^{(\kappa)}}\ |_{\kappa=0}+\frac{x^{0.75}}{\Gamma 1.75}\Delta_{-1}^{-1}\frac{x^{\kappa}}{(1.75)_{-1}^{(\kappa)}}\ |_{\kappa=0}$$
$$+\frac{x^{0}}{\Gamma 1}\Delta_{-1}^{-1}\frac{x^{\kappa}}{(1)_{-1}^{(\kappa)}}\ |_{\kappa=0}$$

which can be arranged as

$$e_{0.25}-e_1(x)=\sum_{r=1}^{3}\frac{x^{(0.25)r}}{\Gamma 1+r(0.25)}\Delta_{-1}^{-1}\frac{x^{\kappa}}{(1+0.25r)_{-1}^{(\kappa)}}\ |_{\kappa=0}.$$

In general, it is easy to obtain (6.9), if $\nu m=1$ *by applying the above method.*

Corollary 6.2.11. *The relation between* $e_{0.35}(x^2)$ *and* $e_1(x^2)$ *is given by*

$$e_{0.5}(x^2)=e_1(x^2)+\frac{x}{\Gamma 1.5}\Delta_{-2}^{-1}\frac{x^{\kappa}}{(1.5)_{-1}^{(\kappa)}}\ |_{\kappa=0}\ . \tag{6.10}$$

Proof: Now, the proof follows by applying the formula $\underset{-2}{\Delta^{-1}}u(\kappa)$*, in the following derivations:*

$e_1(x^2)=1+\frac{x^2}{1!}+\frac{(x^2)^2}{2!}+\frac{(x^2)^3}{3!}+\cdots$

$$e_{0.5}(x^2)=1+\frac{(x^2)^{0.5}}{\Gamma 1.5}+\frac{(x^2)^{1}}{\Gamma 2}+\frac{(x^2)^{1.5}}{\Gamma 2.5}+\frac{(x^2)^{2}}{\Gamma 3}+\cdots$$
$$=1+\frac{x}{\Gamma 1.5}+\frac{x^2}{\Gamma 2}+\frac{x^3}{\Gamma 2.5}+\frac{x^4}{\Gamma 3}+\cdots$$
$$=1+\frac{x^2}{1!}+\frac{(x^2)^2}{2!}+\frac{(x^2)^3}{3!}+\cdots+\frac{x}{\Gamma 1.5}+\frac{x^3}{\Gamma 2.5}+\frac{x^5}{\Gamma 3.5}+\cdots$$
$$=e_1(x^2)+\frac{x}{0.5\Gamma 0.5}+\frac{x^3}{(1.5)(0.5)\Gamma 0.5}+\cdots$$

$$= e_1(x^2) + \frac{x}{\Gamma 1.5} + \frac{x^3}{(1.5)\Gamma(1.5)} + \frac{x^5}{(1.5)(2.5)\Gamma(1.5)} + \cdots$$

$$= e_1(x^2) + \tfrac{x}{\Gamma 1.5}\left[1 + \frac{x^2}{1.5} + \frac{x^4}{(1.5)(2.5)} + \frac{x^6}{(1.5)(2.5)(3.5)} + \cdots\right]$$

$$e_{0.5}(x^2) = e_1(x^2) + \frac{x}{\Gamma 1.5}\Delta_{-2}^{-1}\frac{x^\kappa}{(1.5)_{-1}^{(\kappa)}} \;|_{\kappa=0}.$$

Lemma 6.2.12. *The relation between* $e_1(ix)$ *and* $e_2(ix)$ *is*

$$e_1(ix) - e_2(ix) + i\sin x = 0. \tag{6.11}$$

Proof: $e_2(ix) = 1 + \dfrac{(ix)^2}{2!} + \dfrac{(ix)^4}{4!} + \dfrac{(ix)^6}{6!} + \cdots$

is same as

$$e_2(ix) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x \tag{6.12}$$

$e_1(ix) = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$

can be expressed as $e_1(ix) = e_2(ix) + i\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]$ *Thus,*

$$e_1(ix) - e_2(ix) = i\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right] = -i\sin x \tag{6.13}$$

which completes the proof.

Corollary 6.2.13. *The identity relating* $e_3(ix)$ *and* $e_6(ix)$ *is given by*

$$\frac{-3}{x^3}\left[\frac{d^3}{dx^3}e_3(ix) + ie_6(ix)\right] = \Delta_{-6}^{-1}\frac{x^\kappa(-1)^{\frac{\kappa}{2}}}{4_{-1}^{(\kappa)}} \;|_{\kappa=0}\;. \tag{6.14}$$

Proof: $e_3(ix) = 1 + \dfrac{(ix)^3}{3!} + \dfrac{(ix)^6}{6!} + \dfrac{(ix)^9}{9!} + \cdots$

$$= 1 - \frac{ix^3}{3!} - \frac{x^6}{6!} + i\frac{x^9}{9!} + \frac{x^{12}}{12} + \cdots$$

$$= 1 - \frac{x^6}{6!} + \frac{x^{12}}{12!} - \frac{x^{18}}{18!} + \frac{x^{24}}{24!} + \cdots - i\left[\frac{x^3}{3!} - \frac{x^9}{9!} + \frac{x^{15}}{15!} - \frac{x^{21}}{21!} - \cdots\right]$$

$$\frac{d}{dx}e_3(ix) = \left[0 - \frac{x^5}{5!} + \frac{x^{11}}{11!} - \frac{x^{17}}{17!} + \cdots\right] - i\left[\frac{x^2}{2!} - \frac{x^8}{8!} + \frac{x^{14}}{14!} + \cdots\right]$$
$$\frac{d^2}{dx^2}e_3(ix) = \left[-\frac{x^4}{4!} - \frac{x^{10}}{10!} + \frac{x^{16}}{16!} + \cdots\right] - i\left[\frac{x^1}{1!} - \frac{x^7}{7!} + \frac{x^{13}}{13!} + \cdots\right]$$
$$\frac{d^3}{dx^3}e_3(ix) = -\left[-\frac{x^3}{3!} - \frac{x^9}{9!} + \frac{x^{15}}{15!} + \cdots\right] - i\left[1 - \frac{x^6}{6!} + \frac{x^{12}}{12!} + \cdots\right]$$
$$= -ie_6(ix) - \frac{x^3}{3!}\left[1 - \frac{x^6}{4.5.6.7.8.9} + \frac{x^{15}}{4.5.6.7.8\cdots} + \cdots\right]$$
$$= -ie_6(ix) - \frac{x^3}{3!}\Delta_{-6}^{-1}\frac{x^\kappa(-1)^{\frac{\kappa}{2}}}{4_{-1}^{(\kappa)}}\ |_{\kappa=0},$$

which yields

$$\frac{-3}{x^3}\left[\frac{d^3}{dx^3}e_3(ix) + ie_6(ix)\right] = \Delta_{-6}^{-1}\frac{x^\kappa(-1)^{\frac{\kappa}{2}}}{4_{-1}^{(\kappa)}}\ |_{\kappa=0},$$

and the proof is complete.

Theorem 6.2.14. *Let* $s_1(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ *and* $c_1(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$. *Then* $e_1(x) = c_1(x) + s_1(x)$ *and* $\frac{d}{dx}c_1(x) = s_1(x)$, $\frac{d^2}{dx^2}c_1(x) = \frac{d}{dx}c_1(x) = e_1(x)$.

Proof: From the definition of differentiation, we arrive at

$$\frac{d}{dx}c_1(x) = \lim_{h\to 0}\left[c_1(x+h) - c_1(x)\right]/h$$
$$= \lim_{h\to 0}\left\{1 + \frac{(x+h)^2}{2!} + \frac{(x+h)^4}{4!} + \frac{(x+h)^6}{6!} + \cdots - \left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right]\right\}/h$$
$$= \lim_{h\to 0}\left[\frac{2xh}{2!} + \frac{h^2}{2!} + \frac{4x^3h}{4!} + \frac{6x^5h}{6!}\right]/h,$$

which yields

$$\frac{d}{dx}c_1(x) = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \text{ and } \frac{d^2}{dx^2}c_1(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots,$$

and the proof is complete .

Theorem 6.2.15. *$e_2(x)$ is a solution of the differential equation* $\frac{dy}{dx}+\frac{d^2y}{dx^2}=e(x)$.

Proof: $e_2(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$

$$\frac{d}{dx}e_2(x)=\frac{x^1}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots=s_1(x) \tag{6.15}$$

$$\frac{d^2}{dx^2}e_2(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots=c_1(x) \tag{6.16}$$

Now, the proof follows from (6.15) *and* (6.16).

6.3 Extorial Function in Difference Equation

In this section, we define Extorial function and find the inverse of generalized delta difference on it. Also we arrive at solutions of certain type of higher order difference equation

Definition 6.3.1. *[23] For $-1<\ell<1, \ell\neq 0$ and $\kappa,\nu\in\mathbb{R}$, the ℓ-extorial function, denoted as $e_\nu(\kappa_\ell)$, is defined as*

$$e_\nu(\kappa_\ell)=1+\frac{\kappa_\ell^{(\nu)}}{\Gamma(1\nu+1)}+\frac{\kappa_\ell^{(2\nu)}}{\Gamma(2\nu+1)}+\frac{\kappa_\ell^{(3\nu)}}{\Gamma(3\nu+1)}+\cdots+\infty. \tag{6.17}$$

If $\ell\in(-\infty,\infty), \ell\neq 0$ and κ is a multiple of ℓ and $\nu\in N$, then (6.17) *is defined, and which case all except finite terms of $e_\nu(\kappa_\ell)$ are zero.*

Theorem 6.3.2. *If $1-(1+\ell)^{-1}>0, \kappa\in\mathbb{N}$, then we have*

$$\Delta_\ell^{-\nu}e_1((\ell_1\kappa)_{\ell_1}^{(1)})=\left[1-(1+\ell_1)^{-1}\right]^{-\nu}e_1(\ell_1\kappa)_{\ell_1}^{(1)}. \tag{6.18}$$

Proof: Using Binomial theorem, we have $e_1((\ell\kappa)_\ell^{(1)}) = (1+\ell)^\kappa$.

Taking Δ_ℓ *on both sides, we get*

$\Delta_{\ell_1} e_1((\ell\kappa)_\ell^{(1)}) = (1+\ell)^\kappa - (1+\ell)^{\kappa-1} = (1+\ell)^\kappa(1-(1+\ell)^{-1})$.

which yields $\Delta_\ell^{-1}(e_1(\ell\kappa)_\ell^{(1)}) = [1-(1+\ell)^{-1}]^{-1} e_1(\ell\kappa)_\ell^{(1)}$

Again taking Δ_ℓ^{-1} *on both sides, we get*

$\Delta_\ell^{-2}(e_1(\ell\kappa)_\ell^{(1)}) = [1-(1+\ell)^{-1}]^{-2} e_1(\ell\kappa)_\ell^{(1)}$,

$\Delta_\ell^{-3}(e_1(\ell\kappa)_\ell^{(1)}) = [1-(1+\ell)^{-1}]^{-3} e_1(\ell\kappa)_\ell^{(1)}$

The equation (6.18) follows by continuing the above process and replacing ℓ *by* ℓ_1.

Example 6.3.3. *By replacing* κ *by* m, *we get*

$\Delta_\ell^{-\nu} e_1(\kappa_\ell^{(1)}) = [1-(1+\ell)^{-1}]^{-\nu} e\kappa_\ell^{(1)}, |\ell| < 1$ *and* $\frac{\kappa}{\ell} \in \mathbb{N}$, *and*

$\Delta_\ell^{-\nu}(1+\ell)^{\frac{\kappa}{\ell}} = \left[1-(1+\ell)^{-\ell}\right]^{-\nu}(1+\ell)^{\frac{\kappa}{\ell}}$.

Theorem 6.3.4. *For* $\nu > 0$, $|\frac{\kappa}{\ell} < 1|$, *we have*

$$\Delta_\ell^{-\nu}(1+\ell)^{\frac{\kappa}{\ell}} = (1+\frac{1}{\ell})^\nu(1+\ell)^{\frac{\kappa}{\ell}}. \tag{6.19}$$

Proof: Applying Δ_ℓ *on* $u(\kappa) = (1+\ell)^{\frac{\kappa}{\ell}}$, *we get*

$\Delta_\ell u(\kappa) = (1+\ell)^{\frac{\kappa}{\ell}} - (1+\ell)^{\frac{\kappa-\ell}{\ell}}$ *which yields*

$(1+\ell)^{\frac{\kappa}{\ell}} - (1+\ell)^{\frac{\kappa}{\ell}}(1+\ell)^{-1} = (1+\ell)^{\frac{\kappa}{\ell}}\left[1-\frac{1}{1+\ell}\right] = (1+\ell)^{\frac{\kappa}{\ell}}\frac{\ell}{(1+\ell)}$

$\Delta_\ell^{-1}(1+\ell)^{\frac{\kappa}{\ell}} = (1+\ell)^{\frac{\kappa}{\ell}}\ell^{-1}(1+\ell) = \left(1+\frac{1}{\ell}\right)(1+\ell)^{\frac{\kappa}{\ell}}$ *and*

Similarly, $\Delta_\ell^{-2}(1+\ell)^{\frac{\kappa}{\ell}} = \left(1+\frac{1}{\ell}\right)^2(1+\ell)^{\frac{\kappa}{\ell}}$.

In general, $\Delta_\ell^{-m}(1+\ell)^{\frac{\kappa}{\ell}} = (1+\frac{1}{\ell})^m(1+\ell)^{\frac{\kappa}{\ell}}$.

Now the proof follows by replacing m *by* ν.

Theorem 6.3.5. *$e_2(\kappa_\ell)$ is a solution of the difference equation*

$u(\kappa) + \frac{1}{\ell}\Delta_\ell u(\kappa) = (1+\ell)^{\frac{\kappa}{\ell}}, \frac{\kappa}{\ell} \in \mathbb{N}, \ell \neq 0, |\ell| \leq 1.$

Proof: $e_2(\kappa_\ell) = 1 + \frac{\kappa_\ell^{(2)}}{2!} + \frac{\kappa_\ell^{(4)}}{4!} + \frac{\kappa_\ell^{(6)}}{6!} + \cdots$

Taking Δ_ℓ on both sides, we get

$$\Delta_\ell e_2(\kappa_\ell) = \frac{\ell\kappa_\ell^{(1)}}{1!} + \frac{\ell\kappa_\ell^{(3)}}{3!} + \cdots \tag{6.20}$$

Similarly

$$\frac{1}{\ell}\Delta_\ell e_2(\kappa_\ell) = \frac{\kappa_\ell^{(1)}}{1!} + \frac{\kappa_\ell^{(3)}}{3!} + \cdots \tag{6.21}$$

Adding (6.20) and (6.21), we get $e_2(\kappa_\ell) + \frac{1}{\ell}\Delta_\ell e_2(\kappa_\ell) = e(\kappa_\ell) = (1+\ell)^{\frac{\kappa}{\ell}}$.

Now the proof follows by taking $u(\kappa) = e_2(\kappa_\ell)$.

Theorem 6.3.6. *$e_n(\kappa_\ell)$ is a solution to the difference equation*

$$\sum_{r=0}^{n-1} \frac{1}{\ell^r}\Delta_\ell^r u(\kappa) = e_1(\kappa_\ell). \tag{6.22}$$

Proof: From $\Delta_\ell u(\kappa) = u(\kappa+\ell) - u(\kappa)$ and $\Delta_\ell \kappa_\ell^{(r)} = n\ell\kappa_\ell^{(r-1)}$, we find

$e_n(\kappa_\ell) = 1 + \frac{\kappa_\ell^{(n)}}{n!} + \frac{\kappa_\ell^{(2n)}}{(2n)!} + \frac{\kappa_\ell^{(3n)}}{(3n)!} + \cdots +,$

and hence,

$\Delta_\ell e_n(\kappa_\ell) = \frac{n\ell\kappa_\ell^{(n-1)}}{n!} + \frac{(2n)\ell\kappa_\ell^{(2n-1)}}{(2n)!} + \cdots$

$\frac{1}{\ell}\Delta_\ell e_n(\kappa_\ell) = \frac{\kappa_\ell^{(n-1)}}{(n-1)!} + \frac{\kappa_\ell^{(2n-1)}}{(2n-1)!} + \cdots$

$\frac{1}{\ell^2}\Delta_\ell^2 e_n(\kappa_\ell) = \frac{\kappa_\ell^{(n-1)}}{(n-1)!} + \frac{\kappa_\ell^{(2n-1)}}{(2n-1)!} + \cdots$

$\frac{1}{\ell^{n-1}}\Delta_\ell^{n-1} e_n(\kappa_\ell) = \frac{\kappa_\ell^{(1)}}{(1)!} + \frac{\kappa_\ell^{(n+1)}}{(n+1)!} + \cdots$

Adding the above identities, we get

$e_n(\kappa_\ell) + \frac{1}{\ell}\Delta_\ell e_n(\kappa_\ell) + \cdots + \frac{1}{\ell^{n-1}}\Delta_\ell^{n-1} e_n(\kappa_\ell) = e_1(\kappa_\ell)$. *Finally we get* (6.22).

Corollary 6.3.7. *If κ is integer multiples of ℓ, then $e_n(\kappa_\ell)$ is a solution to the $\ell-$difference equation*

$$\sum_{r=0}^{n-1} \frac{1}{\ell^r}\Delta_\ell^r u(\kappa) = (1+\ell)^{\frac{\kappa}{\ell}}. \tag{6.23}$$

Corollary 6.3.8. *The differential equation* $\sum\limits_{r=0}^{n-1} D^r u(\kappa) = e^\kappa$ *has a solution* $e_n(\kappa) = 1 + \frac{\kappa^n}{n!} + \frac{\kappa^{2n}}{2n!} + \cdots$

Proof: As $\ell \to 0$, in (6.23) *we get the proof.*

Using these functions, we can obtain many results and application in number theory and in calculus.

Chapter 7

2D-Fibonacci Summation with Extorial Functions

7.1 Introduction

In this chapter, second order Fibonacci summation formula of product of polynomial and extorial functions are obtained by two dimensional second order Fibonacci nabla difference operator, its inverse and the two dimensional second order Fibonacci numbers. The function obtained by replacing polynomial into factorial polynomial in the expansion of exponential function is called extorial function. We have mentioned properties of extorial function in the chapter. Here we apply extorial function to two dimensional difference equation.

7.2 2D Nabla Difference Operator

In this section, the definition of 2D Nabla difference operator is defined as follows.

Definition 7.2.1. *Let u be real valued function $a = (a_1, a_2)$ and $\ell = (\ell_1, \ell_2)$ and $(\kappa_1, \kappa_2) \in \mathbb{R}^2$ The Nabla $\ell-$difference operator $\underset{(a)\ell}{\nabla}$ on $u(\kappa_1, \kappa_2)$ is defined as*

$$\underset{(a)\ell}{\nabla} v(\kappa_1, \kappa_2) = v(\kappa_1, \kappa_2) - a_1 v(\kappa_1 + \ell_1, \kappa_2 + \ell_2) - a_2 v(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2). \quad (7.1)$$

and its inverse is given by

$$\underset{(a)\ell}{\nabla} v(\kappa_1, \kappa_2) = u(\kappa_1, \kappa_2) \Rightarrow v(\kappa_1, \kappa_2) = \underset{(a)\ell}{\overset{-1}{\nabla}} u(\kappa_1, \kappa_2) + c, \quad (7.2)$$

where c is an arbitrary constant.

Remark:1 *For our convenient, we denote $E(is\ell) = e((is\ell_1)_{s\ell_1})e((is\ell_2)_{s\ell_2})$, $E(is\kappa) = e((is\kappa_1)_{s\ell_1})e((is\kappa_2)_{s\ell_2})$ and $E(i\kappa) = e((i\kappa_1)_{\ell_1})e((i\kappa_2)_{\ell_2})$*

Example 7.2.2. *Let $1 - \sum_{j=1}^{2} a_j e((j\ell_1)_{\ell_1})e((j\ell_2)_{\ell_2}) \neq 0$. By (7.1) and (7.2), we have*

$$\underset{(a)\ell}{\nabla} E(\kappa) = E(\kappa) - a_1 E(\kappa + \ell) - a_2 E(\kappa + 2\ell). \quad (7.3)$$

$$= E(\kappa)[1 - a_1 E(\ell) - a_2 e(2(\ell_1)_{\ell_1})e(2(\ell_2)_{\ell_2})],$$

which yields

$$\underset{(a)\ell}{\overset{-1}{\nabla}} E(\kappa) = \frac{E(\kappa)}{1 - \sum_{j=1}^{2} a_j e((j\ell_1)_{\ell_1})e((j\ell_2)_{\ell_2})} + c. \quad (7.4)$$

7.3 2D Fibonacci Summation Formula

Here, two dimensional summation formula is obtained by equating closed form and summation form solutions of 2D difference equation.

Theorem 7.3.1. *Let* $\overline{F}_n$ *denotes the* n^{th} *term of the two parameter Fibonacci sequence given in* (6.1) *and* $v(\kappa_1, \kappa_2)$ *be a solution of the equation* $\underset{(a)\ell}{\nabla} v(\kappa_1, \kappa_2) = u(\kappa_1, \kappa_2), (\kappa_1, \kappa_2) \in \mathbb{R}^2$*, then we have*

$$v(\kappa_1, \kappa_2) - (F_n F_{n-1} + a_2 F_{n-1}) v(\kappa_1 + (n+1)\ell_1, \kappa_2 + (n+1)\ell_2)$$

$$- a_2 F_n v(\kappa_1 + (n+2)\ell_1, \kappa_2 + (n+2)\ell_2) = \sum_{i=0}^{n} F_i u(\kappa_1 + i\ell_1, \kappa_2 + i\ell_2). \tag{7.5}$$

Proof. From (7.1) and (7.2), we get

$$v(\kappa_1, \kappa_2) = u(\kappa_1, \kappa_2) + a_1 v(\kappa_1 + \ell_1, \kappa_2 + \ell_2) + a_2 v(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2). \tag{7.6}$$

Replacing κ_1 by $\kappa_1 + \ell_1$, κ_2 by $\kappa_2 + \ell_2$ and then substituting the value $v(\kappa_1 + \ell_1, \kappa_2 + \ell_2)$ in (7.6), we obtain

$$v(\kappa_1, \kappa_2) = u(\kappa_1, \kappa_2) + a_1 u(\kappa_1 + \ell_1, \kappa_2 + \ell_2) + (a_1^2 + a_2) v(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2)$$

$$+ a_1 a_2 v(\kappa_1 + 3\ell_1, \kappa_2 + 3\ell_2). \tag{7.7}$$

Replacing κ_1 by $\kappa_1 + 2\ell_1$, κ_2 by $\kappa_2 + 2\ell_2$ and then substituting the value $v(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2)$ in (7.7), we obtain

$$v(\kappa_1, \kappa_2) = u(\kappa_1, \kappa_2) + a_1 u(\kappa_1 + \ell_1, \kappa_2 + \ell_2) + (a_1^2 + a_2) u(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2)$$

$$+((a_1^2+a_2)a_1+a_1a_2)v(\kappa_1+3\ell_1,\kappa_2+3\ell_2)+(a_1^2+a_2)a_2v(\kappa_1+4\ell_1,\kappa_2+4\ell_2),$$

which can be expressed as

$$v(\kappa_1,\kappa_2)=F_0u(\kappa_1,\kappa_2)+F_1u(\kappa_1+\ell_1,\kappa_2+\ell_2)+F_2u(\kappa_1+2\ell_1,\kappa_2+2\ell_2)$$

$$+(F_2F_1+F_1a_2)v(\kappa_1+3\ell_1,\kappa_2+3\ell_2)+F_2a_2v(\kappa_1+4\ell_1,\kappa_2+4\ell_2)$$

Repeating this process again and again and by induction, we arrive (7.5). □

Corollary 7.3.2. *If* $\sum\limits_{j=1}^{2} a_je((j\ell_1)_{\ell_1})e((j\ell_2)_{\ell_2}) \neq 1$, *then we have*

$$\frac{E(\kappa)-[F_{n-1}F_n+F_{n-1}a_2]e((\kappa_1+(n+1)\ell_1)_{\ell_1})e((\kappa_2+(n+1)\ell_2)_{\ell_2})}{1-\sum\limits_{j=1}^{2} a_je((j\ell_1)_{\ell_1})e((j\ell_2)_{\ell_2})}$$

$$-\frac{F_na_2e((\kappa_2+(n+2)\ell_2)_{\ell_2})e((\kappa_2+(n+1)\ell_2)_{\ell_2})}{1-\sum\limits_{j=1}^{2} a_je((j\ell_1)_{\ell_1})e((j\ell_2)_{\ell_2})}$$

$$=\sum_{i=0}^{n} F_ie((\kappa_1+i\ell_1)_{\ell_1})e((\kappa_2+i\ell_2)_{\ell_2}). \tag{7.8}$$

Proof. Taking $u(\kappa_1,\kappa_2)=E(\kappa)$, and applying (7.4), $\underset{(a)\ell}{\nabla} v(\kappa_1,\kappa_2)=u(\kappa_1,\kappa_2)$,
$v(\kappa_1,\kappa_2)=\underset{(a)\ell}{\overset{-1}{\nabla}} u(\kappa_1,\kappa_2)=\dfrac{E(\kappa)}{1-\sum\limits_{j=1}^{2} a_je((j\ell_1)_{\ell_1})e((j\ell_2)_{\ell_2})}$.
Substituting $v(\kappa_1,\kappa_2)$ and $u(\kappa_1,\kappa_2)$ in (7.5) gives (7.8).

The following example is a numerical verification for (7.8). □

Example 7.3.3. *Taking* $\kappa_1=4,\kappa_2=5,n=2,a_1=2,a_2=2,\ell_1=\ell_2=1$ *in* (7.8)*, and using* $F_0=1,F_1=2,F_2=6$*, we find*

$$\frac{e(4_1)e(5_1)-16e(7_1)e(8_1)-12e(8_1)e(9_1)}{1-2e(1_1)e(1_1)-e(2_1)e(2_1)}=\sum_{i=0}^{2} F_ie((4+i)_1)e((5+i)_1).$$

Applying the extorial values, we arrive

$$\frac{(16)(32) - 12(128)(256) - 5(256)(512)}{-39}$$
$$= F_0 e(4_1)e(5_1) + F_1 e(5_1)e(6_1) + F_2 e(6_1)e(7_1) = 53760.$$

Theorem 7.3.4. *Let* $\sum_{j=1}^{2} a_j E(js\ell) \neq 1$. *Then an exact solution of the 2D difference equation* $\underset{(a:\ell)}{\nabla} v(\kappa_1,\kappa_2) = \kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa)$ *is obtained as*

$$\underset{(a)\ell}{\overset{-1}{\nabla}} \kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa) = \frac{\kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa)}{1 - a_1E(s\ell) - a_2E(2s\ell)} + \sum_{i=0}^{m_1}\sum_{j=0}^{m_2} b_{ij} \underset{(a)\ell}{\overset{-1}{\nabla}} \kappa_1^{m_1-i}\kappa_2^{m_2-j}E(s\kappa) \tag{7.9}$$

where $b_{ij} = \dfrac{\ell_1^i\ell_2^j(a_1E(s\ell) + 2^i2^ja_2E(2s\ell))}{1 - a_1E(s\ell) - a_2E(2s\ell)}$.

Proof. We give proof by induction method. Consider the case $m_1 = 1, m_2 = 0$

By taking $v(\kappa) = \dfrac{\kappa_1\kappa_2^0E(s\kappa)}{1 - a_1E(s\ell) - a_2E(2s\ell)}$ in (7.1), we get

$$\underset{(a)\ell}{\overset{-1}{\nabla}} \kappa_1\kappa_2^0E(s\kappa) = \frac{\kappa_1\kappa_2^0E(s\kappa)}{1 - a_1E(s\ell) - a_2E(2s\ell)} + \frac{\ell_1(a_1E(s\ell) + 2a_2E(2s\ell))}{(1 - a_1E(s\ell) - a_2E(2s\ell))^2} \tag{7.10}$$

Consider the case $m_1 = 0, m_2 = 1$.

Taking $v(\kappa) = \dfrac{\kappa_1^0\kappa_2E(s\kappa)}{1 - a_1E(s\ell) - a_2E(2s\ell)}$ in (7.1) gives

$$\underset{(a)\ell}{\overset{-1}{\nabla}} \kappa_1^0\kappa_2E(s\kappa) = \frac{\kappa_1^0\kappa_2E(s\kappa)}{1 - a_1E(s\ell) - a_2E(2s\ell)} + \frac{\ell_2(a_1E(s\ell) + 2a_2E(2s\ell))}{(1 - a_1E(s\ell) - a_2E(2s\ell))^2}. \tag{7.11}$$

When $m_1 = m_2 = 1$, by taking $v(\kappa) = \dfrac{\kappa_1\kappa_2E(s\kappa)}{1 - a_1E(s\ell) - a_2E(2s\ell)}$ in (7.1), we get

$$\underset{(a)\ell}{\nabla} \frac{\kappa_1\kappa_2E(s\kappa)}{1 - a_1E(s\ell) - a_2E(2s\ell)} = \frac{\kappa_1\kappa_2E(s\kappa)}{1 - a_1E(s\ell) - a_2E(2s\ell)}$$

$$-\frac{a_1(\kappa_1+\ell_1)(\kappa_2+\ell_2)E(s(\kappa+\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}-\frac{a_2(\kappa_1+2\ell_1)(\kappa_2+2\ell_2)E(s(\kappa+\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$=\kappa_1\kappa_2E(s\kappa)-\frac{\kappa_1\ell_2E(s\kappa)(a_1E(s\ell)+2a_2E(2s\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$-\frac{\kappa_2\ell_1E(s\kappa)(a_1E(s\ell)+2a_2E(2s\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}-\frac{\ell_1\ell_2E(s\kappa)(a_1E(s\ell)+4a_2E(2s\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}.$$

Applying $\overset{-1}{\underset{(a)\ell}{\nabla}}$ on both sides, gives

$$\overset{-1}{\underset{(a)\ell}{\nabla}}\kappa_1\kappa_2E(s\kappa)=\frac{\kappa_1\kappa_2E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$+\frac{(\kappa_1\ell_2+\kappa_2\ell_1)E(s\kappa)(a_1E(s\ell)+2a_2E(2s\ell))}{(1-a_1E(s\ell)-a_2E(2s\ell))^2}+\frac{2\ell_1\ell_2E(s\kappa)(a_1E(s\ell)+2a_2E(2s\ell))^2}{(1-a_1E(s\ell)-a_2E(2s\ell))^3}$$

$$+\frac{\ell_1\ell_2E(s\kappa)(a_1E(s\ell)+4a_2E(2s\ell))}{(1-a_1E(s\ell)-a_2E(2s\ell))^2}. \tag{7.12}$$

When $m_1=2, m_2=0$, by taking $v(\kappa)=\dfrac{\kappa_1^2\kappa_2^0E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}$ in (7.1), we get

$$\underset{(a)\ell}{\nabla}\frac{\kappa_1^2\kappa_2^0E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}=\frac{\kappa_1^2\kappa_2^0E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$-\frac{a_1(\kappa_1+\ell_1)^2(\kappa_2+\ell_2)^0e((s\kappa_1+\ell_1)_{s\ell_1})e((s\kappa_2+\ell_2)_{s\ell_2})}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$-\frac{a_2(\kappa_1+2\ell_1)^2(\kappa_2+2\ell_2)^0e((s\kappa_1+\ell_1)_{s\ell_1})e((s\kappa_2+\ell_2)_{s\ell_2})}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$=\kappa_1^2E(s\kappa)-\frac{2\kappa_1\ell_1E(s\kappa)(a_1E(s\ell)+2a_2E(2s\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$-\frac{\kappa_2\ell_1E(s\kappa)(a_1E(s\ell)+2a_2E(2s\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$-\frac{\ell_1^2E(s\kappa)(a_1E(s\ell)+4a_2E(2s\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}.$$

Applying $\overset{-1}{\nabla}$ on both sides, we get

$$\overset{-1}{\underset{(a)\ell}{\nabla}} \kappa_1^2 E(s\kappa) = \frac{\kappa_1^2 E(s\kappa)}{1 - a_1 E(s\ell) - a_2 E(2s\ell)} + \frac{2\kappa_1 \ell_1 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}$$
$$+ \frac{2\ell_1^2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))^2}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^3} + \frac{\ell_1^2 E(s\kappa)(a_1 E(s\ell) + 4a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}. \quad (7.13)$$

Similarly,$m_1 = 0, m_2 = 2$, by taking $v(\kappa) = \dfrac{\kappa_1^0 \kappa_2^2 E(s\kappa)}{1 - a_1 E(s\ell) - a_2 E(2s\ell)}$ in (7.1) gives

$$\overset{-1}{\underset{(a)\ell}{\nabla}} \kappa_2^2 E(s\kappa) = \frac{\kappa_2^2 E(s\kappa)}{1 - a_1 E(s\ell) - a_2 E(2s\ell)} + \frac{2\kappa_2 \ell_2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}$$
$$+ \frac{2\ell_2^2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))^2}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^3} + \frac{\ell_2^2 E(s\kappa)(a_1 E(s\ell) + 4a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}. \quad (7.14)$$

When $m_1 = 2, m_2 = 1$,by taking $v(\kappa) = \dfrac{\kappa_1^2 \kappa_2 E(s\kappa)}{1 - a_1 E(s\ell) - a_2 E(2s\ell)}$ in (7.1), gives

$$\overset{-1}{\underset{(a)\ell}{\nabla}} \kappa_1^2 \kappa_2 E(s\kappa) = \frac{\kappa_1^2 \kappa_2 E(s\kappa)}{1 - a_1 E(s\ell) - a_2 E(2s\ell)} + \frac{2\kappa_1 \kappa_2 \ell_1 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}$$
$$+ \frac{4\kappa_1 \ell_1 \ell_2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))^2}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^3} + \frac{2\kappa_2 \ell_1^2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))^2}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^3}$$
$$+ \frac{6\ell_1^2 \ell_2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))^3}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^4} + \frac{\kappa_2 \ell_1^2 E(s\kappa)(a_1 E(s\ell) + 4a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}$$
$$+ \frac{\kappa_1^2 \ell_2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2} + \frac{2\kappa_1 \ell_1 \ell_2 E(s\kappa)(a_1 E(s\ell) + 4a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}$$
$$+ \frac{6\ell_1^2 \ell_2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))(a_1 E(s\ell) + 4a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^3}$$
$$+ \frac{\ell_1^2 \ell_2 E(s\kappa)(a_1 E(s\ell) + 8a_2 E(2s\ell))^3}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}. \quad (7.15)$$

Similarly,$m_1 = 1, m_2 = 2$, by taking $v(\kappa) = \dfrac{\kappa_1 \kappa_2^2 E(s\kappa)}{1 - a_1 E(s\ell) - a_2 E(2s\ell)}$ in (7.1), we get

$$\overset{-1}{\underset{(a)\ell}{\nabla}} \kappa_1 \kappa_2^2 E(s\kappa) = \frac{\kappa_1 \kappa_2^2 E(s\kappa)}{1 - a_1 E(s\ell) - a_2 E(2s\ell)} + \frac{2\kappa_1 \kappa_2 \ell_2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}$$
$$+ \frac{4\kappa_2 \ell_1 \ell_2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))^2}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^3} + \frac{2\kappa_1 \ell_2^2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))^2}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^3}$$
$$+ \frac{6\ell_1 \ell_2^2 E(s\kappa)(a_1 E(s\ell) + 2a_2 E(2s\ell))^3}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^4} + \frac{\kappa_1 \ell_2^2 E(s\kappa)(a_1 E(s\ell) + 4a_2 E(2s\ell))}{(1 - a_1 E(s\ell) - a_2 E(2s\ell))^2}$$

$$+\frac{6\ell_1^2\ell_2E(s\kappa)(a_1E(s\ell)+2a_2E(2s\ell))(a_1E(s\ell)+4a_2E(2s\ell))}{(1-a_1E(s\ell)-a_2E(2s\ell))^3}$$

$$+\frac{\kappa_2^2\ell_1E(s\kappa)(a_1E(s\ell)+2a_2E(2s\ell))}{(1-a_1E(s\ell)-a_2E(2s\ell))^2}+\frac{2\kappa_2\ell_1\ell_2E(s\kappa)(a_1E(s\ell)+4a_2E(2s\ell))}{(1-a_1E(s\ell)-a_2E(2s\ell))^2}$$

$$+\frac{6\ell_1\ell_2^2E(s\kappa)(a_1E(s\ell)+2a_2E(2s\ell))(a_1E(s\ell)+4a_2E(2s\ell))}{(1-a_1E(s\ell)-a_2E(2s\ell))^3}$$

$$+\frac{\ell_1\ell_2^2E(s\kappa)(a_1E(s\ell)+8a_2E(2s\ell))}{(1-a_1E(s\ell)-a_2E(2s\ell))^2} \tag{7.16}$$

When $m_1=2, m_2=2$, by taking $v(\kappa)=\dfrac{\kappa_1^2\kappa_2^2E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}$ in (7.1), we get

$$\underset{(a)\ell}{\nabla}\frac{\kappa_1^2\kappa_2^2E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}=\frac{\kappa_1^2\kappa_2^2E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$-\frac{a_1(\kappa_1+\ell_1)^2(\kappa_2+\ell_2)^2E(s(\kappa+\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}-\frac{a_2(\kappa_1+2\ell_1)^2(\kappa_2+2\ell_2)^2E(s(\kappa+\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

Applying $\overset{-1}{\underset{(a)\ell}{\nabla}}$ on both sides, we get

In general, by taking $v(\kappa)=\dfrac{\kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}$ in (7.1)

$$\underset{(a)\ell}{\nabla}\frac{\kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}=\frac{\kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$-\frac{a_1(\kappa_1+\ell_1)^{m_1}(\kappa_2+\ell_2)^{m_2}E(s(\kappa+\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}-\frac{a_2(\kappa_1+2\ell_1)^{m_1}(\kappa_2+2\ell_2)^{m_2}E(s(\kappa+\ell))}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$=\frac{\kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}-\frac{a_1\sum\limits_{r_1=1}^{m_1}m_1c_{r_1}\kappa_1^{m_1-r_1}\ell_1^{r_1}\sum\limits_{r_2=1}^{m_2}m_2c_{r_2}\kappa_2^{m_2-r_2}\ell_2^{r_2}E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

$$-\frac{a_2\sum\limits_{r_1=1}^{m_1}m_1c_{r_1}\kappa_1^{m_1-r_1}(2\ell_1)^{r_1}\sum\limits_{r_2=1}^{m_2}m_2c_{r_2}\kappa_2^{m_2-r_2}(2\ell_2)^{r_2}E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)}$$

By expanding the term, we get

$$\underset{(a)\ell}{\nabla} \frac{\kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa)}{1-a_1E(s\ell)-a_2E(2s\ell)} = b_{00}\kappa_1^{m_1}\kappa_2^{m_2}+b_{01}\kappa_1^{m_1}\kappa_2^{m_2-1}+b_{02}\kappa_1^{m_1}\kappa_2^{m_2-2}$$

$$+\cdots+b_{0m_2}\kappa_1^{m_1}\kappa_2^{0}+b_{10}\kappa_1^{m_1-1}\kappa_2^{m_2}+\cdots+b_{m_10}\kappa_1^{0}\kappa_2^{m_2}$$

By iteration ,we can find (7.9). □

7.4 2D-Fibonacci Summation for Factorials and Extorials

In this section, by equating clossed and summation form, we are able to obtain two dimensional summation formula related to factorial and extorial function.

Theorem 7.4.1. *If* $v(\kappa) = \underset{(a)\ell}{\overset{-1}{\nabla}} \kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa)$ *is an exact solution of* (7.9)*, then*

$$\underset{(a)\ell}{\overset{-1}{\nabla}} \kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa) - (F_nF_{n-1}+a_2F_{n-1}) \underset{(a)\ell}{\overset{-1}{\nabla}} (\kappa_1+(n+1)\ell_1)^{m_1}$$

$$(\kappa_2+(n+1)\ell_2)^{m_2}e((s(\kappa_1+(n+1))_{s\ell_1})e((s(\kappa_2+(n+1))_{s\ell_2})$$

$$-a_2F_n \underset{(a)\ell}{\overset{-1}{\nabla}} (\kappa_1+(n+2)\ell_1)^{m_1}(\kappa_2+(n+2)\ell_2)^{m_2}e((s(\kappa_1+(n+2))_{s\ell_1})e((s(\kappa_2+(n+2))_{s\ell_2})$$

$$= \sum_{i=0}^{n} F_i(\kappa_1+i\ell_1)^{m_1}(\kappa_2+i\ell_2)^{m_2}e(s(\kappa_1+i\ell_1)_{s\ell_1})e(s(\kappa_2+i\ell_2)_{s\ell_2}). \qquad (7.17)$$

Proof. Taking $u(\kappa_1,\kappa_2) = \kappa_1^{m_1}\kappa_2^{m_2}E(s\kappa)$ in Theorem 7.3.1, we get (7.17). □

Example 7.4.2. *The following is an example for verification of* (7.17)*. If* $\kappa_1 = 4$, $\kappa_2 = 2$, $n = 2$,$m_2 = 0$ *and* $m_1, a_1, a_2, \ell_1, \ell_2$ *are 1, the* (7.17) *becomes*

$$\underset{(1,1)}{\overset{-1}{\nabla}} 4e(4_1)e(2_1) - 3 \underset{(1,1)}{\overset{-1}{\nabla}} 7e(7_1)e(5_1) - 2 \underset{(1,1)}{\overset{-1}{\nabla}} 8e(8_1)e(6_1)$$

$$= \sum_{i=0}^{2} F_i(4+i)e((4+i)_1)e((2+i)_1) = 13824.$$

Theorem 7.4.3. *Let $u(\kappa_1, \kappa_2)$ and $v(\kappa_1, \kappa_2)$ be the real valued function. Then,*

$$\overset{-1}{\underset{(a:\ell)}{\nabla}} \left[u\left(\kappa_1, \kappa_2\right) v\left(\kappa_1, \kappa_2\right)\right] = u\left(\kappa_1, \kappa_2\right) \overset{-1}{\underset{(a:\ell)}{\nabla}} v\left(\kappa_1, \kappa_2\right)$$
$$- a_1 \overset{-1}{\underset{(a:\ell)}{\nabla}} \left[\overset{-1}{\underset{(a:\ell)}{\nabla}} v\left(\kappa_1 + \ell_1, \kappa_2 + \ell_2\right) \underset{(1,0:\ell)}{\nabla} u\left(\kappa_1, \kappa_2\right)\right]$$
$$- a_2 \overset{-1}{\underset{(a:\ell)}{\nabla}} \left[\overset{-1}{\underset{(a:\ell)}{\nabla}} v\left(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2\right) \underset{(0,1:\ell)}{\nabla} u\left(\kappa_1, \kappa_2\right)\right]. \quad (7.18)$$

Proof. Taking $v\left(\kappa_1, \kappa_2\right) = u\left(\kappa_1, \kappa_2\right) w\left(\kappa_1, \kappa_2\right)$ in (7.1), we get

$$\underset{(a:\ell)}{\nabla} \left[u\left(\kappa_1, \kappa_2\right) w\left(\kappa_1, \kappa_2\right)\right] = u\left(\kappa_1, \kappa_2\right) w\left(\kappa_1, \kappa_2\right)$$
$$- a_1 u\left(\kappa_1 + \ell_1, \kappa_2 + \ell_2\right) w\left(\kappa_1 + \ell_1, \kappa_2 + \ell_2\right)$$
$$- a_2 u\left(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2\right) w\left(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2\right)$$

Adding and Subtracting $a_1 u\left(\kappa_1, \kappa_2\right) w\left(\kappa_1 + \ell_1, \kappa_2 + \ell_2\right)$ and

$a_2 u\left(\kappa_1, \kappa_2\right) w\left(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2\right)$ yields

$$\underset{(a:\ell)}{\nabla} [u(\kappa_1, \kappa_2) w(\kappa_1, \kappa_2)] = u(\kappa_1, \kappa_2) \underset{(a:\ell)}{\nabla} w(\kappa_1, \kappa_2) + a_1 w(\kappa_1 + \ell_1, \kappa_2 + \ell_2) \underset{(1,0:\ell)}{\nabla} u(\kappa_1, \kappa_2)$$
$$+ a_2 w(\kappa_1 + 2\ell_1, \kappa_2 + 2\ell_2) \underset{(0,1:\ell)}{\nabla} u(\kappa_1, \kappa_2)$$

taking $w(\kappa_1, \kappa_2) = \overset{-1}{\underset{(a:\ell)}{\nabla}} v(\kappa_1, \kappa_2)$ and applying $\overset{-1}{\underset{(a:\ell)}{\nabla}}$ on both sides, we get (7.18). □

Remark:2 For similarity, we denote $E(i\ell) = e((i\ell_1)_{\ell_1})e((i\ell_2)_{\ell_2})$ and

$$E(i\kappa) = e((i\kappa_1)_{\ell_1})e((i\kappa_2)_{\ell_2})$$

Corollary 7.4.4. *Let $u(\kappa_1,\kappa_2)$ and $v(\kappa_1,\kappa_2)$ be the real valued functions. Then,*

$$\underset{(a:\ell)}{\overset{-1}{\nabla}} u(\kappa_1,\kappa_2)E(\kappa) = u(\kappa_1,\kappa_2)\underset{(a:\ell)}{\overset{-1}{\nabla}} E(\kappa) - a_1 \underset{(a:\ell)}{\overset{-1}{\nabla}}\left[\underset{(a:\ell)}{\overset{-1}{\nabla}} E(\kappa+\ell)\underset{(1,0:\ell)}{\nabla} u(\kappa_1,\kappa_2)\right] - a_2 \underset{(a:\ell)}{\overset{-1}{\nabla}}\left[\underset{(a:\ell)}{\overset{-1}{\nabla}} E(\kappa+2\ell)\underset{(0,1:\ell)}{\nabla} u(\kappa_1,\kappa_2)\right]. \tag{7.19}$$

Proof. Taking $v(\kappa_1,\kappa_2) = E(\kappa)$ in (7.18), yields(7.19) □

Corollary 7.4.5. *An exact solution of the 2D second order diffference equation $\underset{a(\ell)}{\nabla} v(\kappa_1,\kappa_2) = \kappa_1^2\kappa_2^2E(\kappa)$ is given by*

$$\begin{aligned}
\underset{(a:\ell)}{\overset{-1}{\nabla}} \kappa_1^2\kappa_2^2E(\kappa) &= \frac{\kappa_1^2\kappa_2^2E(\kappa)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{a_1\ell_2^2\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1^2E(\kappa+\ell)}{1-a_1E(\ell)-a_2E(2\ell)}\\
&+\frac{2a_1\ell_2\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1^2\kappa_2E(\kappa+\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{a_1\ell_1^2\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_2^2E(\kappa+\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{a_1\ell_1^2\ell_2^2\underset{(a:\ell)}{\overset{-1}{\nabla}}E(\kappa+\ell)}{1-a_1E(\ell)-a_2E(2\ell)}\\
&+\frac{2\ell_1^2\ell_2a_1\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_2E(\kappa+\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{2a_1\ell_1\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1\kappa_2^2E(\kappa+\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{2\ell_1\ell_2^2a_1\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1E(\kappa+\ell)}{1-a_1E(\ell)-a_2E(2\ell)}\\
&+\frac{4\ell_1\ell_2a_1\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_2\kappa_1E(\kappa+\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{4a_2\ell_2^2\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1^2E(\kappa+2\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{4a_2\ell_2\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1^2\kappa_2E(\kappa+2\ell)}{1-a_1E(\ell)-a_2E(2\ell)}\\
&+\frac{4a_2\ell_1^2\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_2^2E(\kappa+2\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{16a_2\ell_1^2\ell_2^2\underset{(a:\ell)}{\overset{-1}{\nabla}}E(\kappa+2\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{16\ell_1^2\ell_2a_2\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_2E(\kappa+2\ell)}{1-a_1E(\ell)-a_2E(2\ell)}\\
&+\frac{4a_2\ell_1\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1\kappa_2^2E(\kappa+2\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{16\ell_1\ell_2^2a_2\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1E(\kappa+2\ell)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{16\ell_1\ell_2a_2\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_2\kappa_1E(\kappa+2\ell)}{1-a_1E(\ell)-a_2E(2\ell)}.
\end{aligned} \tag{7.20}$$

Proof. Taking $u(\kappa_1,\kappa_2) = \kappa_1\kappa_2^0$ in (7.19) and using (7.4), we find

$$\mathop{\nabla}\limits_{(a:\ell)}^{-1} \kappa_1\kappa_2^0 E(\kappa) = \frac{\kappa_1\kappa_2^0 E(\kappa)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{\ell_1 a_1 E(\kappa+\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2} + \frac{2\ell_1 a_2 E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2}. \quad (7.21)$$

Similarly, we can find

$$\mathop{\nabla}\limits_{(a:\ell)}^{-1} \kappa_1^0\kappa_2 E(\kappa) = \frac{\kappa_1^0\kappa_2 E(\kappa)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{\ell_2 a_1 E(\kappa+\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2} + \frac{2\ell_2 a_2 E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2}. \quad (7.22)$$

Taking $u(\kappa_1,\kappa_2)=\kappa_1\kappa_2$ in (7.19) and using (7.4), we get

$$\mathop{\nabla}\limits_{(a:\ell)}^{-1} \kappa_1\kappa_2 E(\kappa) = \frac{\kappa_1\kappa_2 E(\kappa)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{a_1(\kappa_1\ell_2+\kappa_2\ell_1+\ell_1\ell_2)E(\kappa+\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2} + \frac{2a_1^2\ell_1\ell_2 E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3} + \frac{8a_1a_2\ell_1\ell_2 E(\kappa+3\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3} + \frac{2a_2(\kappa_1\ell_2+\kappa_2\ell_1+2\ell_1\ell_2)E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2} + \frac{8a_2^2\ell_1\ell_2 E(\kappa+4\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3} \quad (7.23)$$

Taking $u(\kappa_1,\kappa_2)=\kappa_1^2\kappa_2^0$ in (7.19) and using (7.4) , we arrive

$$\mathop{\nabla}\limits_{(a:\ell)}^{-1} \kappa_1^2\kappa_2^0 E(\kappa) = \frac{\kappa_1^2\kappa_2^0 E(\kappa)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{2\kappa_1 a_1\ell_1 E(\kappa+\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2} + \frac{2a_1^2\ell_1^2 E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3} + \frac{8a_1a_2\ell_1^2 E(\kappa+3\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3} + \frac{a_1\ell_1^2 E(\kappa+\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2} + \frac{4\kappa_1 a_1\ell_1 a_2 E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2} + \frac{8a_2^2\ell_1^2 E(\kappa+4\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3} + \frac{4a_2\ell_1^2 e((\kappa_1+2\ell_1)_{\ell_1})e((\kappa_2+2\ell2)_{\ell_2})}{(1-a_1E(\ell)-a_2E(2\ell))^2}. \quad (7.24)$$

Taking $u(\kappa_1,\kappa_2)=\kappa_1^0\kappa_2^2$ in (7.19) and using (7.4), we obtain

$$\mathop{\nabla}\limits_{(a:\ell)}^{-1} \kappa_1^0\kappa_2 E(\kappa) = \frac{\kappa_1^0\kappa_2^2 E(\kappa)}{1-a_1E(\ell)-a_2E(2\ell)} + \frac{2\kappa_2 a_1\ell_2 E(\kappa+\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2}$$

$$+\frac{2a_1^2\ell_2^2E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3}+\frac{8a_1a_2\ell_2^2E(\kappa+3\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3}$$
$$+\frac{a_1\ell_2^2E(\kappa+\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2}+\frac{4\kappa_2a_2\ell_2a_2E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2}$$
$$+\frac{8a_2^2\ell_2^2E(\kappa+4\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3}+\frac{4a_2\ell_2^2e((\kappa_1+2\ell_1)_{\ell_1})e((\kappa_2+2\ell 2)_{\ell_2})}{(1-a_1E(\ell)-a_2E(2\ell))^2}. \tag{7.25}$$

Taking $u(\kappa_1,\kappa_2)=\kappa_1^2\kappa_2$ in (7.19) and using (7.4), we get

$$\overset{-1}{\underset{(a:\ell)}{\nabla}}\kappa_1^2\kappa_2E(\kappa)=\frac{\kappa_1^2\kappa_2E(\kappa)}{1-a_1E(\ell)-a_2E(2\ell)}$$
$$+\frac{a_1[\kappa_1^2\ell_2+\kappa_2\ell_1^2+\ell_1^2\ell_2+2\ell_1\kappa_1\kappa_2+2\kappa_1\ell_1\ell_2]E(\kappa+\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2}$$
$$+\frac{2a_1^2\ell_1[2\kappa_1\ell_2+\kappa_2\ell_1+3\ell_1\ell_2]E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3}$$
$$+\frac{2a_2[\kappa_1^2\ell_2+2\kappa_2\ell_1^2+4\ell_1^2\ell_2+2\ell_1\kappa_1\kappa_2+4\kappa_1\ell_1\ell_2]E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2}$$
$$+\frac{6a_1^3\ell_1^2\ell_2E(\kappa+3\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^4}+\frac{4a_1a_2\ell_1[2\kappa_2\ell_1+4\kappa_1\ell_2+9\ell_1\ell_2]E(\kappa+3\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3}$$
$$+\frac{36a_1^2a_2\ell_1^2\ell_2E(\kappa+4\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^4}+\frac{8a_2^2\ell_1[2\kappa_1\ell_2+\kappa_2\ell_1+6\ell_1\ell_2]E(\kappa+4\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3}$$
$$+\frac{72a_2^2a_1\ell_1^2\ell_2e((\kappa_1+5\ell_1)_{\ell_1})e((\kappa_2+5\ell_2)_{\ell_2})}{(1-a_1E(\ell)-a_2E(2\ell))^4}+\frac{48a_2^3\ell_1^2\ell_2e((\kappa_1+6\ell_1)_{\ell_1})e((\kappa_2+6\ell_2)_{\ell_2})}{(1-a_1E(\ell)-a_2E(2\ell))^4}. \tag{7.26}$$

Taking $u(\kappa_1,\kappa_2)=\kappa_1\kappa_2^2$ in (7.19) and using (7.4), we derive

$$\overset{-1}{\underset{(a:\ell)}{\nabla}}\kappa_1\kappa_2^2E(\kappa)=\frac{\kappa_1\kappa_2^2E(\kappa)}{1-a_1E(\ell)-a_2E(2\ell)}$$
$$+\frac{a_1[\kappa_2^2\ell_1+\kappa_1\ell_2^2+\ell_1\ell_2^2+2\ell_2\kappa_1\kappa_2+2\kappa_2\ell_1\ell_2]E(\kappa+\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2}$$
$$+\frac{2a_1^2\ell_2[2\kappa_2\ell_1+\kappa_1\ell_2+3\ell_1\ell_2]E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3}$$

$$+\frac{2a_2[\kappa_2^2\ell_1+2\kappa_1\ell_2^2+4\ell_1\ell_2^2+2\ell_2\kappa_1\kappa_2+4\kappa_2\ell_1\ell_2]E(\kappa+2\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^2}$$

$$+\frac{6a_1^3\ell_1\ell_2^2E(\kappa+3\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^4}+\frac{4a_1a_2\ell_2[2\kappa_1\ell_2+4\kappa_2\ell_1+9\ell_1\ell_2]E(\kappa+3\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3}$$

$$+\frac{36a_1^2a_2\ell_1\ell_2^2E(\kappa+4\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^4}+\frac{8a_2^2\ell_2[2\kappa_2\ell_1+\kappa_1\ell_2+6\ell_1\ell_2]E(\kappa+4\ell)}{(1-a_1E(\ell)-a_2E(2\ell))^3}$$

$$+\frac{72a_1a_2^2\ell_1\ell_2^2e((\kappa_1+5\ell_1)_{\ell_1})e((\kappa_2+5\ell_2)_{\ell_2})}{(1-a_1E(\ell)-a_2E(2\ell))^4}+\frac{48a_2^3\ell_1\ell_2^2e((\kappa_1+6\ell_1)_{\ell_1})e((\kappa_2+6\ell_2)_{\ell_2})}{(1-a_1E(\ell)-a_2E(2\ell))^4}. \tag{7.27}$$

By taking $u(\kappa_1,\kappa_2)=\kappa_1^2\kappa_2^2$ in (7.19) and using (7.4), we get (7.20).

□

Corollary 7.4.6. *If* $v(\kappa_1,\kappa_2)=\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1^2\kappa_2^2E(\kappa)$ *is an exact solution given by* (7.20), *then the 2D-Fibonacci summation formula for* $\kappa_1^2\kappa_2^2E(\kappa)$ *is*

$$\sum_{i=0}^{n}F_i(\kappa+i\ell)^2e((\kappa+i\ell)_\ell)=\underset{(a:\ell)}{\overset{-1}{\nabla}}\kappa_1^2\kappa_2^2E(\kappa)$$
$$-(F_nF_{n-1}+a_2F_{n-1})\underset{(a\ell)}{\overset{-1}{\nabla}}(\kappa_1+(n+1))^2(\kappa_2+(n+1))^2e((\kappa_1+(n+1)\ell_1)_{\ell_1})e((\kappa_2+(n+1)\ell_2)_{\ell_2})$$
$$-a_2F_n\underset{(a\ell)}{\overset{-1}{\nabla}}(\kappa_1+(n+2))^2(\kappa_2+(n+2))^2e((\kappa_1+(n+2)\ell_1)_{\ell_1})e((\kappa_2+(n+2)\ell_2)_{\ell_2}). \tag{7.28}$$

Proof. By taking $u(\kappa_1,\kappa_2)=\kappa_1^2\kappa_2^2E(\kappa)$ in Theorem 7.3.1, we get (7.28) □

Example 7.4.7. *Let* $\kappa_1=\kappa_2=a_1=a_2=\ell_1=\ell_2=1$, $n=2$ *in* (7.28)*. Then*

$$\underset{(1:1)}{\overset{-1}{\nabla}}e(1_1)e(1_1)-3\underset{(1:1)}{\overset{-1}{\nabla}}4(4)^2e(4_1)e(4_1)-2\underset{(1:1)}{\overset{-1}{\nabla}}5(5)^2e(5_1)e(5_1)$$
$$=\sum_{i=0}^{2}F_i(1+i)(1+i)^2e((1+i)_1)e((1+i)_1)=3588.$$

The concept of two dimensional difference operator and its difference equations are used to obtain solutions of wave equation, which is already mentioned in the final chapter.

Chapter 8

Extorial in RL Circuits and Heat Flows

8.1 Exact Solutions of RL Circuits

The resistor and inductor are the most fundamental linear (element having linear relationship between voltage and current) and passive elements in electric circuits. When resistor and inductor are connected across voltage supply, the circuit so obtained is called as RL circuit which can be either in a series or parallel circuit depending on the nature of connection between the resistor and inductor. The extorial function act as exact solution of difference equation arrived for current flow in RL circuits. Here, we obtain exact solutions for difference equation of RL circuits.

8.2 Current Flows in RL Circuit

Consider a RL circuit by using the Kirchhoff's circuit rule. The differential equation connecting voltage V, resistance R, current I and induction L in series is given by first order linear difference equation

$$V = RI(\kappa) + L\frac{dI(\kappa)}{d\kappa}. \tag{8.1}$$

The discrete analogue of (8.1) is assumed by replacing $dI(\kappa) = \Delta I(\kappa)$, where $\Delta I(\kappa) = I(\kappa+1) - I(\kappa)$ and $d\kappa = 1$ in (8.1).

The corresponding difference equation for the current flows in RL series circuit in the discrete case takes the form, at time κ

$$v(\kappa) = RI(\kappa) + L\Delta I(\kappa). \tag{8.2}$$

Due to the resistance of conductor, heat temperature may be raised in the RL circuit. In that case, we need to modify the difference equation (8.2). In that case the difference equation (8.2) becomes fractional difference equation as

$$V = RI(\kappa) + L\Delta^{\nu} I(\kappa), (0 < \nu < 1). \tag{8.3}$$

Which can be expressed as

$$\frac{V(\kappa) - RI(\kappa)}{L} = \Delta^{\nu} I(\kappa)$$

. The corresponding discrete integral equations

$$\Delta^{-\nu}\left(\frac{V(\kappa) - RI(\kappa)}{L}\right) = I(\kappa)$$

. By applying γ^{th} order delta sum given by (3.1),

$$\frac{1}{L\Gamma(\nu)}\sum_{s=0}^{\kappa-\nu}\frac{\Gamma(\kappa-s)}{\Gamma(\kappa-s-(\nu-1))}(V(s)-RI(s))=I(\kappa), \tag{8.4}$$

The corresponding fractional difference equation for de-energizing in RL circuit is obtained by putting $v(\kappa)=0$. In this case, we get

$$0=RI(\kappa)+L\Delta^{\nu}I(\kappa). \tag{8.5}$$

Which is the same as

$$\Delta^{\nu}I(\kappa)=-\frac{RI(\kappa)}{L}.$$

$$I(\kappa)=\Delta^{-\nu}\left(-\frac{RI(\kappa)}{L}\right)=-\frac{R}{L}\Delta^{-\nu}I(\kappa)$$

By applying fractional order delta integration (3.1), we obtain

$$I(\kappa)=\frac{R}{L\Gamma(\nu)}\sum_{s=0}^{\kappa-\nu}\frac{\Gamma(\kappa-s)}{\Gamma(\kappa-s-(\nu-1))}I(s), \tag{8.6}$$

The solution (8.4) and (8.6) are summation forms. Through our research, we identifies that these fractional difference equations have exact type solutions, when the initial time a is taken as zero. We obtain exact solution for the equations (8.2) and (8.3) using our newly defined extorial functions.

8.3 Extorial Type Solution of RL Circuit

In this section, we find solution of equation (8.2) after arriving at some basic results of extorial functions. This extorial function is easily obtained by replacing polynomial κ^n into factorial polynomial in the expansion of exponential function e^κ. This function is useful to arrive at solutions for fractional difference equation. Formal definition and some properties of extorials are given in chapter 4.

Consider the extorial function $e_1((m\kappa)_{(m)})$ is defined by

$$e_1((m\kappa)_{(m)}) = 1 + \frac{(m\kappa)_m^{(1)}}{1!} + \frac{(m\kappa)_m^{(2)}}{2!} + \cdots + \infty = \sum_{r=0}^{\infty} \frac{(m\kappa)_m^{(r)}}{r!}, \tag{8.7}$$

where $(m\kappa)_m^{(r)} = (m\kappa)(m\kappa - m)\cdots(m\kappa - (r-1)m)$ for positive integer r, is a falling polynomial factorial. In general, for real index ν, we have

$$e_{(\nu)}((m\kappa)_{(m)}) = 1 + \frac{(m\kappa)_m^{(\nu)}}{1!} + \frac{(m\kappa)_m^{(2\nu)}}{2!} + \cdots + \infty = \sum_{r=0}^{\infty} \frac{(m\kappa)_m^{(r\nu)}}{r!}, \tag{8.8}$$

where $(m\kappa)_m^{(r\nu)} = (m)^{(r\nu)} \dfrac{\Gamma(\kappa+1)}{\Gamma(K+1-r\nu)}$ and $\Gamma(.)$ is the gamma function.

Lemma 8.3.1. *If* $e_1((m\kappa)_{(m)})$ *is an extorial function, then we have*

$$\Delta e_1((m\kappa)_{(m)}) = (m)e_1((m\kappa)_{(m)}). \tag{8.9}$$

Proof. Applying Δ on the extorial function $e_1((m\kappa)_{(m)})$, we arrive at

$$\begin{aligned}
\Delta e_1((m\kappa)_{(m)}) &= \Delta(1) + \Delta\frac{(m\kappa)_m^{(1)}}{1!} + \Delta\frac{(m\kappa)_m^{(2)}}{2!} + \cdots + \infty \\
&= 0 + \frac{(m(\kappa+1))_m^{(1)}}{1!} + \frac{(m\kappa)_m^{(1)}}{1!} + \frac{1}{2!}\Big[(m(\kappa+1))_m^{(2)} - (m\kappa)_m^{(2)}\Big] + \cdots + \infty \\
&= \frac{(m)}{1!} + \frac{1}{2!}\Big[(m\kappa+m)(m\kappa) - (m\kappa)(m\kappa-m)\Big] + \cdots + \infty \\
&= m + \frac{2m}{2!}(m\kappa)_m^{(1)} + \frac{3m}{3!}(m\kappa)_m^{(2)} + \cdots + \infty \\
&= m\Big[1 + \frac{(m\kappa)_m^{(\kappa)}}{1!} + \frac{(m\kappa)_m^{(2)}}{2!} + \cdots + \infty\Big]
\end{aligned}$$

$\Delta e^{(m\kappa)_m} = (m)e_1((m\kappa)_{(m)})$. □

Lemma 8.3.2. *The extorial function* $u(\kappa) = e_1((m\kappa)_{(m)})$ *is a solution of equation*

$$\Big(A\Delta^2 + B\Delta + C\Big)u(\kappa) = 0, \tag{8.10}$$

if m *is a root of the auxiliary equation* $Am^2 + Bm + C = 0$.

Proof. If we try $u(\kappa) = e_1((m\kappa)_{(m)})$ as a solution of equation(8.10), then it should satisfy the equation

$$A\Delta^2 e_1((m\kappa)_{(m)}) + B\Delta e_1((m\kappa)_{(m)}) + Ce_1((m\kappa)_{(m)}) = 0. \tag{8.11}$$

By linear property of Δ and the expansion of $e_1((m\kappa)_{(m)})$, we arrive at

$$\Delta_\ell e_1((m\kappa)_{(m)}) = 0 + (m)\frac{(m\kappa)^{(0)}_{(m)}}{1!} + \frac{2m(m\kappa)^{(1)}_{(m)}}{2!} + \frac{3m(m\kappa)^{(2)}_{(m)}}{3!} + ...$$

$$\text{i.e, } \Delta_\ell e_1((m\kappa)_{(m)}) = m\Big[1 + \frac{(m\kappa)^{(1)}_{(m)}}{1!} + \frac{(m\kappa)^{(2)}_{(m)}}{2!} + ...\Big] = me_1((m\kappa)_{(m)}),$$

which yields

$$\Delta^2_\ell e_1((m\kappa)_{(m)}) = (m)\Delta_\ell e_1((m\kappa)_{(m)}) = (m)^2 e_1((m\kappa)_{(m)}).$$

Applying the values of $\Delta e_1((m\kappa)_{(m)})$ and $\Delta^2_\ell e_1((m\kappa)_{(m)})$ in (8.11), we obtain

$$\Big(Am^2 + Bm + C\Big)e_1((m\kappa)_{(m)}) = 0,$$

Since $e_1((m\kappa)_m) \neq 0$, we get

$$Am^2 + Bm + C = 0. \tag{8.12}$$

Hence $u(\kappa) = e_1((m\kappa)_{(m)})$ is a solution of (8.10) when m is a root of (8.11). □

Remark 8.3.3. *The above lemma can be extended to higher order linear difference equation with constant coefficients.*

Theorem 8.3.4. *Let I_0 be initial value of $I(\kappa)$ and $\nu = 1$. The de-energizing difference equation (8.5) for $\nu = 1$ has a solution of the form*

$$I(\kappa) = I_0 e_1\left(\left(\frac{-R}{L}\kappa\right)_{(\frac{-R}{L})}\right) \tag{8.13}$$

where e_1 denotes the extorial function.

Proof. Consider the first order difference equation $L\Delta I(\kappa) + RI(\kappa) = 0$ which is obtained from (8.5) by taking $\nu = 1$ and its Auxillary equation ML+R=0. The auxiliary equation $mL + R = 0$ has an unique solution $m = \dfrac{-R}{L}$, $L \neq 0$. Applying Lemma 8.3.2 for first order difference equation by taking $A = 0$, $I(t) = I_0 e_1((\frac{-R}{L}\kappa)_{(\frac{-R}{L})})$, which is a solution of the equation (8.5) for $\nu = 1$. □

Theorem 8.3.5. *For $\nu = 1$, the energizing difference equation (8.3) has a solution*

$$I(\kappa) = \frac{V}{L(e^s - 1) + R} + I_0 e_1\left(\left(\frac{-R}{L}\kappa\right)_{(\frac{-R}{L})}\right), \tag{8.14}$$

where s is a constant.

Proof. Let $I(\kappa) = \dfrac{V}{c}e^{s\kappa}$ be a solution of equation (8.3) for $\nu = 1$, where c is to be determined. Since s is a constant, we get $\Delta e^{s\kappa} = e^{s(\kappa+1)} - e^{s\kappa} = (e^s - 1)e^{s\kappa}$ and $\Delta I(\kappa) = \dfrac{V}{c}\Delta e^{s\kappa} = \dfrac{V}{c}e^{s\kappa}(e^s - 1)$.

Substituting $\nu = 1$, $I(\kappa)$ and $\Delta I(\kappa)$ in (8.3), we arrive

$$I(\kappa)R + L\Delta I(\kappa) = R\frac{V}{c}e^{s\kappa} + L\Big[\frac{V}{c}e^{s\kappa}(e^s - 1)\Big],$$

which yields

$\Big[R + L\Delta\Big] I = \frac{V}{c}\Big[L(e^s - 1 + R)\Big]e^{s\kappa}.$

Hence, taking $c = L(e^s - 1 + R)$, we find a particular solution of equation (8.3) when $\nu = 1$, as

$$I(\kappa) = \frac{V}{L(e^s - 1 + R)} e^{s\kappa}. \tag{8.15}$$

Now (8.14) follows by adding (8.13) and (8.15) the proof is complete. □

Corollary 8.3.6. *If* $I_0 = \frac{-V}{R}$, *then the extorial solution of difference equation* (8.2) *of the RL circuit is* $I(\kappa) = \frac{V}{R} - \frac{V}{R} e_1\left(\left(\frac{-R}{L}\kappa\right)_{\left(\frac{-R}{L}\right)}\right).$

Proof. The proof follows by taking $s = 0$ in (8.14). □

Finding the solutions of integer order difference equation is comparatively easier than the fractional order difference equation.

8.4 Extorial Energizing for RL Circuit

In this section, we derive at the solution of RL circuit model with extorial energizing. Here, we deal with fractional order difference equation also.

Theorem 8.4.1. *The flow of current in the RL circuit creates chaos due to increase of temperature of heat. In this case, the difference equation of RL circuit becomes*

$$Ve_1((s\kappa)_s) = RI(\kappa) + L\Delta^{\nu} I(\kappa), (0 < \nu < 1). \tag{8.16}$$

Equation (8.16) is ν^{th} order fractional difference equation. When there is no choas in RL circuit, the parameter ν takes integer value.

Theorem 8.4.2. *For $\nu = 1$, the difference equation (8.16) has a extorial solution*

$$I(\kappa) = \frac{Ve_1((s\kappa)_{s\ell})}{L(e^s - 1) + R} + I_0 e_1\Big(\Big(\frac{-R}{L}\kappa\Big)_{(\frac{-R}{L})}\Big). \tag{8.17}$$

Proof. Let $I(\kappa) = \dfrac{V}{c}e_1((s\kappa)_s)$ be a solution of equation (8.16) $(\nu = 1)$, where c is to be determined. Since s is a constant, from (8.17), we get

$\Delta e_1((s\kappa)_s) = e_1((s(\kappa+1))_{(s)}) - e_1((s\kappa)_s)$ which gives

$$\Delta I(\kappa) = \frac{V}{c}\Delta e_1((s\kappa)_s) = \frac{V}{c}e_1((s\kappa)_{(s)})(e_1((1)_{(s)}) - 1).$$

Substituting $I(\kappa)$ and $\Delta I(\kappa)$ in the above equation, we arrive

$$I(\kappa)R + L\Delta I(\kappa) = R\frac{V}{c}e_1((s\kappa)_s) + L\Big[\frac{V}{c}e^{(s\kappa)}(e_1(1_{(s)}) - 1)\Big],$$

which yields $\Big[R + L\Delta\Big]I = \dfrac{V}{c}\Big[L(e_1(\ell_{(s)}) - 1 + R)\Big]e_1((s\kappa)_s)$.

Hence taking $c = L(e_1(1_{(s)}) - 1 + R)$, we find $I(\kappa) = \dfrac{V}{L(e_1(1_{(s)}) - 1 + R)}e^{s\kappa}$ is a particular solution of equation when $\nu = 1$ and (8.17) follows. □

Theorem 8.4.3. *For $0 < \nu < 1$, the energizing fractional difference equation*

$$Ve_1((s\kappa)_{(s)}) = I(\kappa)R + \Delta^{\nu} I(\kappa), \tag{8.18}$$

has an extorial solution of the form

$$\frac{Ve_1((s\kappa)_{(s)})}{L(e_1(\ell_{(s)}) - 1)^{\nu} + R} + I_0 e_1\Big(\Big(\frac{-R}{L}\Big)^{\frac{1}{\nu}}\kappa_{(\frac{-R}{L})^{\frac{1}{\nu}}}\Big). \tag{8.19}$$

Proof. We try $I(\kappa) = Vce_1((s\kappa)_{(s)})$ as a solution of equation (8.18), where c is to be determined.

$\Delta I(\kappa) = Vc(e_1(1_{(s)}) - 1)e_1((s\kappa)_{(s)})$, $\Delta^2 I(\kappa) = Vc(e_1(1_{(s)}) - 1)^2 e_1((s\kappa)_{(s)}) \cdots$,

$\Delta^\nu I(\kappa) = Vc(e_1(1_{(s)}) - 1)^\nu e_1((s\kappa)_{(s)})$ is obtained from $\Delta I(\kappa) = I(\kappa+1) - I(\kappa)$.

Substituting I and $\Delta^\nu I$ in (8.18), we find

$$I(\kappa)R + \frac{L}{\ell}\Delta^\nu I(\kappa) = RVce_1((s\kappa)_{(s)}) + L\Big[Vc(e_1(1_{(s)}) - 1)^\nu e_1((s\kappa)_{(s)})\Big]$$
$$= Vc\Big[\frac{L}{\ell}(e_1(1_{(s)}) - 1)^\nu + R\Big]e_1((s\kappa)_{(s)}).$$

Hence $I(\kappa) = \dfrac{V}{L(e_1(1_{(s)}) - 1)^\nu + R}e_1((s\kappa)_{(s)})$ is a particular solution of (8.19). □

Thus extorial function is used to obtain the solution to RL-circuit difference equation . Also we have obtained solution of RL circuit of chaos situation represented by fractional order difference equation.

8.5 Fractional Difference Heat Equation Model

In this section, we apply the alpha and Fibonacci difference operators and obtain new model of heat equations. The solution of these equations are expressed in terms of extorial functions. The materials up to three dimensions i.e., rod, thin plate and medium are taken for study and the transfer of heat is examined. The two operators (alpha and Fibonacci) are used for the study of transfer of heat and are defined accordingly.

Let $\alpha \neq 0$, $l = (1, 1, 1, ..., 1)$, $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n) \in \mathbb{R}^n$ and $v(\kappa)$ be a real valued n-variable function defined on $\mathbb{R}^n$. The n-variable α-difference operator, denoted as Δ_α, on $v(\kappa)$ is defined by

$$\underset{\alpha}{\Delta} v(\kappa) = v(\kappa_1 + 1, \kappa_2 + 1, ..., \kappa_n + 1) - \alpha v(\kappa_1, \kappa_2, ..., \kappa_n). \tag{8.20}$$

This operator becomes partial α-difference operator if we replace by $\kappa_i + 1$ in centain component i. Thus the above definition of the alpha and Fibonacci difference operators and its equations are employed in the forthcoming sections and solutions are derived for heat equations.

Also we present solutions of partial fractional alpha difference equation with polynomial factorial and extorial functions. We also apply these type of solutions to heat flows. In the following lemma, some identities related to alpha difference operator on extorial function are given.

Lemma 8.5.1. *Let $\kappa^{(rn)} \neq 0$, $n \in N$. Then we have the following identities with extorial function:*

(i). $\Delta_\alpha e_1(\kappa) = e_1(\kappa)[1 + 1 - \alpha]$,

(ii). $\Delta_\alpha e_{(-1)}(\kappa) = e_{(-1)}(\kappa)[e_{(-1)} - \alpha]^{(-1)}$,

(iii). $\Delta_\alpha e_1((-\kappa)) = e_1((-\kappa))[1 + 1 - \alpha]$, $\kappa > 0$.

Proof. (i). By (8.20), and applying Δ_α on $e_1(\kappa)$, we arrive

$\Delta_\alpha e_1(\kappa) = e_1((\kappa + 1)) - \alpha e_1(\kappa) = e_1(\kappa).e_1 - \alpha e_1(\kappa) = e_1(\kappa)[e_1(1) - \alpha]$

$$= e_1(\kappa)[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots - \alpha] = e_1(\kappa)[1 + 1 - \alpha].$$

(ii). By (8.20), and applying Δ_α on $e(\kappa^{(-1)})$, we arrive

$$\Delta_\alpha e_{(-1)}(\kappa) = e_{(-1)}((\kappa + 1)) - \alpha e_{(-1)}(\kappa) = e_{(-1)}(\kappa).e_{(-1)} - \alpha e_{(-1)}(\kappa).$$

$$= e_{(-1)}(\kappa)[e_{(-1)} - \alpha]^{(-1)}.$$

(iii). follows from (ii) by replacing κ as $-\kappa$. □

Theorem 8.5.2. *If* $v(\kappa_1, \kappa_2) = e_1((\kappa_1)).e_1((\kappa_2))$ *then we have the identities:*

$(i)\Delta_\alpha v(\kappa_1, \kappa_2) = e_1((\kappa_1)).e_1((\kappa_2))\Big[e_1(1) - \alpha\Big]$,

$(ii)\Delta_\alpha v(\kappa_1, \kappa_2) = e_1((\kappa_2)).e_1((\kappa_1))\Big[e_1((1)) - \alpha\Big]$.

Proof. $(i)\Delta_\alpha v(\kappa_1, \kappa_2) = e_1((\kappa_1))\Big[\Delta_\alpha e_1((\kappa_2))\Big]$

$$= e_1((\kappa_1))\Big[e_1((\kappa_2 + 1)) - \alpha e_1((\kappa_2))\Big]$$

$$= e_1((\kappa_1))e_1((\kappa_2))\Big[e_1(1) - \alpha\Big].$$

In the similar way, the proof of (ii) follows. □

Assume that $v(\kappa_1, \kappa_2)$ be the temperature of a rod at position κ_1 at time κ_2, ℓ_1 and ℓ_2 be shift values of κ_1 and κ_2 respectively and γ be the rate of conductivity of rod. When considering impact of external climate change on the rod, the partial α - difference equation of heat flow in the rod becomes fractional α- difference equation

$$\underset{\alpha}{\Delta^\nu} v(\kappa_1, \kappa_2) = \gamma\Big[\underset{\alpha}{\Delta^\nu} v(\kappa_1, \kappa_2) + \underset{\alpha}{\Delta^\nu} v(\kappa_1, \kappa_2)\Big]. \tag{8.21}$$

Theorem 8.5.3. *If* $\gamma = \Big[e_1(1) - \alpha/e_1(\pm(1)) - \alpha\Big]$, *then the function* $v(\kappa_1, \kappa_2) = e_1((\kappa_1)).e_1((\kappa_2))$ *is the exact solution of the* α*- difference equation* (8.21).

Proof. By applying the Theorem 8.5.2 , we get the proof. □

Corollary 8.5.4. *The fractional partial* α*-difference heat equation* (8.21) *has a solution of the form*

$v(\kappa_1, \kappa_2) = e_1((\kappa_1)).e_1((\kappa_2))$ *if* $\gamma = \Big[(e_1(1) - \alpha)^\nu/(e_1(\pm(1)) - \alpha)^\nu\Big]$.

Assume that $v(\kappa_1, \kappa_2, \kappa_3)$ be the temperature of a thin plate at position (κ_1, κ_2) at time κ_3. Let (1,1,1) be the shift values of (κ_1, κ_2) and κ_3 and γ be the rate of conductivity of thin plate. The fractional partial α-difference heat equation of thin plate is given by

$$\underset{\alpha}{\Delta^\nu} v(\kappa_1, \kappa_2, \kappa_3) = \gamma \left\{ \underset{\alpha}{\Delta^\nu} v(\kappa_1, \kappa_2, \kappa_3) + \underset{\alpha}{\Delta^\nu} v(\kappa_1, \kappa_2, \kappa_3) \right\}. \tag{8.22}$$

Corollary 8.5.5. *If* $\gamma = \Big[1 + 1 - \alpha)^\nu/(e_1(\pm(1)_1) - \alpha)^\nu\Big]$, *then the function* $v(\kappa) = \prod_{i=1}^{3} e_1(\kappa_{i(1_i)})$ *is an exact solution of the fractional partial heat equation* (8.22).

Assume that $v(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$ be the temperature of a medium at position $(\kappa_1, \kappa_2, \kappa_3)$ at time κ_4 and at density κ_5. Let (1,1,1,1,1) be the shift values of $(\kappa_1, \kappa_2, \kappa_3)$, κ_4 and κ_5 and γ be the rate of conductivity of medium. The fractional partial α-difference equation of heat flow in medium is

$$\underset{\alpha}{\Delta^\nu} v(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5) = \gamma \left\{ \underset{\alpha(\pm 1)}{\Delta^\nu} v(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5) \right\}. \tag{8.23}$$

Corollary 8.5.6. *If* $\gamma = \Big[1 + 1 - \alpha)^{\nu} + e_1(1 + 1 - \alpha)^{\nu}/(e_1(\pm(1)_1) - \alpha)^{\nu}\Big]$, *then* $v(\kappa) = \prod_{i=1}^{5} e_1(\kappa_{i(1_i)})$ *is a closed form solution of the fractional partial* α*-difference equation* (8.23).

The extorial function is used to obtain solution to heat equation. Also it is used to obtain fractional order partial difference equation. Thus the extorial function plays vital role in the fractional calculus and yields several applications.

CONCLUSION

In this book entitled "*Extorial Function and Its Applications Using Delta Operators*", we have attempted to derive a new type of extorial function with the inclusion of polynomial factorial with two applications. We also improve the already existing function related to delta operator, alpha delta operator, Fibonacci delta operator and fractional order delta operator and obtain effective applications in RL circuits and Heat equation.

Here, various difference operators mentioned above have been employed in the study of this extorial function. Basic definition and preliminary lemmas of above said operators and its inverse, Bernoulli's polynomials, striling numbers, Riemann zeta factorial function, summation formula, numerical and exact solution of fractional order difference equation, have been applied for getting main results. Also summation and closed (exact) form solution of difference equations for certain type RL circuits and heat equations have been obtained by using extorial function and inverse of delta operators. Here the exponential function is extended to extorial function by substituting falling and raising factorials.

This extorial function satisfies certain type fractional and higher difference equations. Here, when $\ell \to 0$, the difference equation becomes differential equation and the model is simple for solving differential equations. Also fractional real order extorial function is used to obtain solutions of fractional real order difference equation. This concept generates applications in the field of physical sciences, fractional calculus and Numerical methods.

Finally, we want to acknowledge that the theory, results and applications are originally derived in a unique way. The results incorporated in this book have been published in various referred international journal.An innovative attempt has been initiated to make a solution for fractional order difference equation by extorial and Riemann zeta factorial functions with applications in RL circuit and Heat flow.

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